14 One-Loop QCD Corrections

Until now our calculations have been limited for the most part to the lowest perturbative order or 'tree' graphs. But all processes receive higher-order contributions – usually called radiative corrections – from diagrams that contain 'loops', even those (such as flavor changing neutral reactions) for which the lowest-order tree amplitudes are absent. Then new effects can only arise from loops, some typical examples are the K^0–$\overline{K}^0$ mixing and the effective $\Delta S = 1$ neutral current (penguin) considered in Chap. 11. Weak decays offer an excellent opportunity for the study of radiative corrections due to either QCD or electroweak interactions. An n-loop diagram has an implicit factor $(\hbar)^n$; a tree (zero-loop) diagram has $(\hbar)^0$. The $(\hbar)^n$ factor shows that the tree graphs are equivalent to classical (Born) approximation, whereas loops are synonymous with quantum effects. This chapter introduces some important concepts and basic calculational methods of quantum corrections.

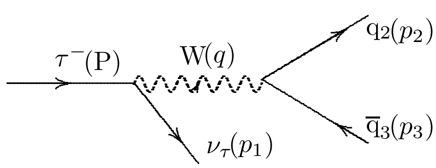

Fig. 14.1. Tree diagram of weak decay $\tau^- \to \nu_\tau + q_2 + \overline{q}_3$

As a first illustration, let us consider the one-loop QCD corrections to the inclusive semileptonic decay of the lepton τ as described by $\tau \to \nu_\tau +$ quark pairs. This mode has been studied in Sect. 13.4 at the tree diagram level (Fig. 14.1). The QCD corrections to order $g_s^2 = 4\pi\alpha_s$ are represented by the five diagrams in Fig. 14.2 and Fig. 14.3.

Although the diagrams of Fig. 14.2 are plagued with *ultraviolet* divergences due to high momenta of virtual particles in the loop integrals, these divergences will be removed by the *renormalization* procedure, which ultimately yields a finite (renormalized) amplitude.

To order α_s, the radiatively corrected rate of $\tau \to \nu_\tau + q_2 + \overline{q}_3$ has two parts. One, Γ_{Vi} with virtual gluons, comes from the interference between the tree amplitude (Fig. 14.1) and the renormalized amplitude (Fig. 14.2). The other, Γ_{Re} with real gluons, comes from the square of the bremsstrahlung amplitude (Fig. 14.3). Both Γ_{Vi} and Γ_{Re} are still ill defined because of another kind of divergence, viz. the *infrared* divergence, which originates from low momenta of virtual or real massless gluons. What is well defined is the sum $\Gamma_{\mathrm{Vi}} + \Gamma_{\mathrm{Re}}$, in which the infrared divergences cancel each other. This chapter shows how to deal with both ultraviolet (UV) and infrared (IR) divergences encountered in quantum corrections and how to calculate the finite parts. We remark that five diagrams similar to those considered here also represent the α_s-order QCD corrections to the $e^+ + e^- \rightarrow$ hadrons cross-section, discussed in (13.5) and (13.65). The only difference is that a photon (connecting the e^+e^- with the quark pair) replaces the W boson.

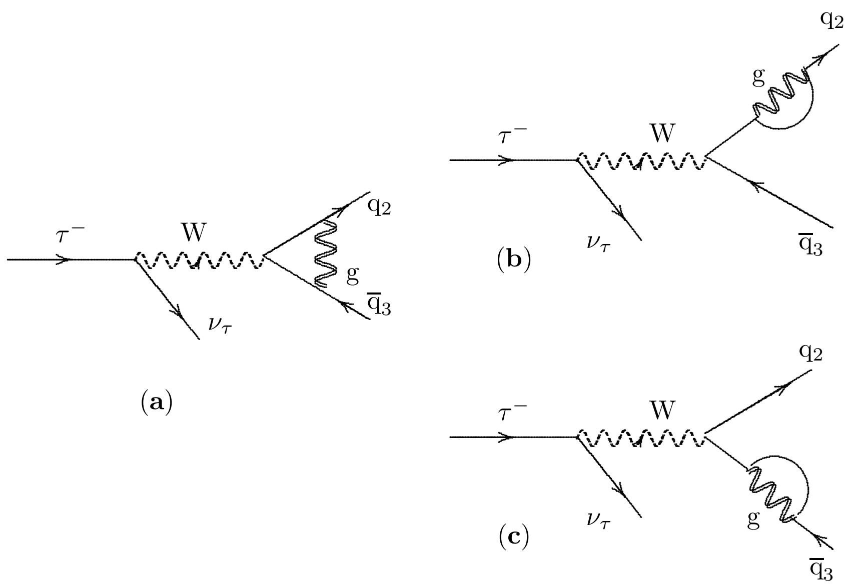

Fig. 14.2a–c. Virtual gluon corrections to weak decay $\tau^- \rightarrow \nu_\tau + q_2 + \overline{q}_3$

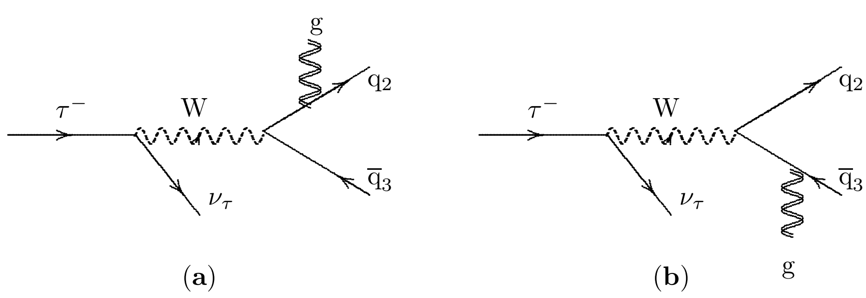

Fig. 14.3a, b. Real gluon emission in weak decay $\tau^- \rightarrow \nu_\tau + q_2 + \overline{q}_3 + g$

Before computing these QCD radiative corrections, we first evaluate the vertex function $\Gamma^\mu(p_2, p_3)$ and the quark self-energy $\Sigma(p)$ which represent the essential parts of the diagrams in Fig. 14.2.

14.1 Vertex Function

To zero order of the QCD coupling constant g_s (Fig. 14.4a), the weak interaction vertex $\mathrm{W}(q) \to \mathrm{q}_2(p_2) + \overline{\mathrm{q}}_3(p_3)$ can be written as $\gamma^\mu(1-\gamma_5)$ to be inserted between the spinors $\overline{u}(p_2)$ and $v(p_3)$. The weak coupling constant $g_\mathrm{W} = -\mathrm{i}\, g\, V_{\mathrm{q}_2\mathrm{q}_3}/2\sqrt{2}$ is implicit. When effects of virtual gluons to order g_s^2 (Fig. 14.4b) are considered, the tree-level weak vertex $\gamma^\mu(1-\gamma_5)$ becomes the QCD-corrected weak vertex function $\Gamma^\mu(p_2, p_3)$. Thus

$$\overline{u}(p_2)\,\gamma^\mu(1-\gamma_5)v(p_3) \longrightarrow \overline{u}(p_2)\,[\gamma^\mu(1-\gamma_5) + \Gamma^\mu(p_2,p_3)]\,v(p_3)\ , \tag{14.1}$$

where $\Gamma^\mu(p_2, p_3)$ has the following expression from Feynman rules:

$$\begin{aligned}\Gamma^\mu(p_2,p_3) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}(-\mathrm{i}\,g_\mathrm{s}\gamma^\rho \tfrac{\lambda_j}{2})\frac{\mathrm{i}}{\not k + \not p_2 - m + \mathrm{i}\eta}\gamma^\mu(1-\gamma_5)\frac{\mathrm{i}}{\not k - \not p_3 - m + \mathrm{i}\eta} \\ \times(-\mathrm{i}\,g_\mathrm{s}\gamma^\sigma \tfrac{\lambda_j}{2})\frac{\mathrm{i}\,[-g_{\rho\sigma} + (1-\xi)k_\rho k_\sigma/k^2]}{k^2+\mathrm{i}\eta}\ .\end{aligned} \tag{14.2}$$

For simplicity, the quark masses are taken to be equal ($m_2 = m_3 = m$).

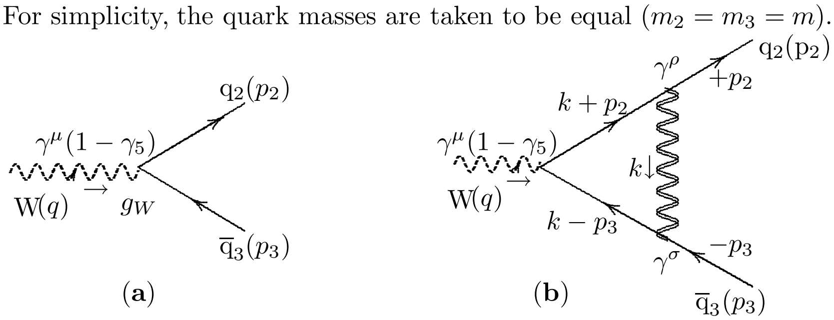

Fig. 14.4. (**a**) Bare vertex $\gamma^\mu(1-\gamma_5)$; (**b**) dressed vertex function $\Gamma^\mu(p_2,p_3)$

The dependence of $\Gamma^\mu(p_2, p_3)$ on the gauge-fixing term ξ of the gluon propagator in fact will be canceled by the dependence of δ_q on ξ [this quantity δ_q which comes from the self-energy term $\Sigma(p)$ of Fig. 14.2b, c, will be introduced later in (40)]. Therefore, when we sum over the three diagrams of Fig. 14.2 to obtain the *ultraviolet-convergent renormalized vertex function*, the ξ-dependent contributions are canceled so that we can use the Feynman–'t Hooft gauge ($\xi = 1$) from the outset. For SU(N_c) color group, using (7.45) and (7.46), the sum over j of $(\frac{\lambda_j}{2})(\frac{\lambda_j}{2})$ in (2) yields

$$\sum_j \frac{\lambda_j}{2}\frac{\lambda_j}{2} = C_2(N_\mathrm{c}) = \frac{N_\mathrm{c}^2-1}{2N_\mathrm{c}}\ , \tag{14.3}$$

which is equal to $\frac{4}{3}$ for $N_\mathrm{c} = 3$. All of these operations lead to

$$\Gamma^\mu(p_2,p_3) = \frac{-4\mathrm{i}g_\mathrm{s}^2}{3}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\gamma^\rho(\not k + \not p_2 + m)\gamma^\mu(1-\gamma_5)(\not k - \not p_3 + m)\gamma_\rho}{(k^2+2k\cdot p_2 + \mathrm{i}\eta)(k^2 - 2k\cdot p_3 + \mathrm{i}\eta)(k^2+\mathrm{i}\eta)}\,. \tag{14.4}$$

When $\Gamma^\mu(p_2, p_3)$ is inserted between $\overline{u}(p_2)$ and $v(p_3)$ and the Dirac equation $\overline{u}(p_2)(\not p_2 - m) = 0 = (\not p_3 + m)v(p_3)$ together with $\not a \not b + \not b \not a = 2a \cdot b$ are used, the numerator $\gamma^\rho(\not k + \not p_2 + m)\gamma^\mu(1-\gamma_5)(\not k - \not p_3 + m)\gamma_\rho$ of (4) can be rewritten as $\mathcal{N}^\mu(k, p_2, p_3)$ defined by

$$\mathcal{N}^\mu(k, p_2, p_3) \equiv \Big\{[-4p_2 \cdot p_3 + 4k \cdot (p_2 - p_3)]\gamma^\mu + 4mk^\mu - 4 \not k (p_2 - p_3)^\mu + \gamma^\rho \not k \gamma^\mu \not k \gamma_\rho\Big\}(1-\gamma_5) + 4\, m \not k \gamma^\mu \gamma_5 \ . \tag{14.5}$$

As we will see, most of the integrals in loop diagrams diverge, we must cut off or regularize the momentum variable to handle the infinities encountered in the integrations. There are at least two methods: the Pauli–Villars covariant momentum cutoff (Problem 14.5), and the *dimensional regularization*. The latter, invented by 't Hooft and Veltman, also independently by Bollini and Giambiagi and by Ashmore, is used here. No matter which regularization method is adopted, any choice is as good as any other, provided that the symmetries of the theory (e.g. Lorentz and gauge invariances) are preserved throughout the calculations.

The idea of dimensional regularization is simple, and is based on the observation that the degree of divergences of loop integrals may be decreased by lowering the space-time dimension. Dimensional regularization amounts to computing divergent integrals in an arbitrary space-time dimension n smaller than 4, resulting in a finite result, as long as the parameter $\varepsilon \equiv 4 - n$ is nonvanishing. Then the result is analytically continued to $n \geq 4$. The original infinities in the physical $n = 4$ world then arise as poles at ε, in the form of $\Gamma(2 - \frac{n}{2}) \sim 2/\varepsilon$, where $\Gamma(x)$ is the Euler function. Appropriate formulas are given in the Appendix. The advantage of this scheme is that Feynman rules and the gauge symmetry of the theory do not depend on n.

Remark. While there is no problem with the extension of $\gamma^\mu = g^{\mu\nu}\gamma_\nu$ to arbitrary n dimensions (the metric tensor $g^{\mu\nu}$ associated with $\{\gamma^\mu, \gamma^\nu\}$ can be defined for $\mu, \nu = 0, \cdots n-1$), the generalization of $\gamma_5 = \gamma^5$ to $n \neq 4$

$$\gamma_5 = \mathrm{i}\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = -\mathrm{i}\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 = \frac{\mathrm{i}}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma = \frac{\mathrm{i}}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\,\gamma_\nu\,\gamma_\rho\,\gamma_\sigma,$$

does present a problem, since the antisymmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ is only defined in 4 dimensions. One may still define the equivalent of γ_5 in n dimensions as

$$\widetilde{\gamma} = \frac{\mathrm{i}}{4!}\varepsilon_{\mu_1\cdots\mu_n}\gamma^{\mu_1}\cdots\gamma^{\mu_n} \ . \tag{14.6}$$

However, when taking the trace of products of γ_μ matrices and $\widetilde{\gamma}$, one will get into trouble. For instance, the trace $\mathrm{Tr}[\gamma_\mu\,\gamma_\nu\,\gamma_\rho\,\gamma_\sigma\widetilde{\gamma}]$ which gives $4\mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma} \neq 0$ for $n = 4$, would vanish if we use (6) for $n \neq 4$. One might worry about the presence of γ_5 in our calculation. For the problem considered here, γ_5

is innocuous. The trace of $\gamma_\mu\gamma_\nu\cdots\gamma_5$ never occurs in our integral loops, it arises only in the computation of the rate. However, for the latter, it is even unnecessary to perform the trace with γ_5 because we first integrate over the symmetric phase space $\int \mathrm{d}\boldsymbol{p}_2\,\mathrm{d}\boldsymbol{p}_3$, such that the contributions of γ_5 vanish (they are antisymmetric in $\boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_3$). We will explicitly show that γ_5 is harmless in the course of the calculation.

Furthermore, the n-dimensional Dirac matrices obey $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 2^{n/2}g^{\mu\nu}$ (n even) or $2^{n-2}g^{\mu\nu}$ (n odd), however, the n dependence of $2^{n/2}$ or 2^{n-2} does not affect the momentum integrals, these factors can be safely replaced by 4, their value at $n \to 4$, when we compute physical quantities. As a result[1], the traces of Dirac matrices *without* the γ_5 depend only on their anticommutation relations, and can be safely taken from the corresponding four-dimensional formulas, for instance $\mathrm{Tr}(\not{a}\,\not{b}) = 4\,a\cdot b$. However, $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = n$. ■

With (5), the $\Gamma^\mu(p_2,p_3)$ becomes in n space-time dimensions

$$\Gamma^\mu(p_2,p_3) = \frac{-4\mathrm{i}g_\mathrm{s}^2}{3}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathcal{N}^\mu(k,p_2,p_3)}{(k^2+2k\cdot p_2)(k^2-2k\cdot p_3)(k^2)}\ . \tag{14.7}$$

In (7) and what follows, the small $+\mathrm{i}\eta$ in the denominator of (4) is omitted, although we keep in mind that the Feynman prescription $+\mathrm{i}\eta$ in the propagators is important for the evaluation of the $\mathrm{d}^n k$ integrals, using the Wick rotation (Appendix). On the second line of (5), the first term $\gamma^\rho\ \not{k}\gamma^\mu\ \not{k}\gamma_\rho$, which is quadratic in k makes the integral divergent as $k\to\infty$. Therefore, we keep $\gamma^\rho\ \not{k}\gamma^\mu\ \not{k}\gamma_\rho = (2-n)\ \not{k}\gamma^\mu\ \not{k}$ in n dimensions (Appendix), instead of $-2\ \not{k}\gamma^\mu\ \not{k}$, when $n=4$. The difference between the $(2-n)$ and -2 of the $\not{k}\gamma^\mu\ \not{k}$ term, when multiplied by the pole $1/\varepsilon$, yields finite terms as $\varepsilon\to 0$.

From now on, we adopt a convenient method due to Feynman which consists in writing every product in the denominator of a product of propagators as an integral over auxiliary variables (Appendix). Accordingly, the denominator of (7) can be written as

$$\frac{1}{(k^2+2k\cdot p_2)(k^2-2k\cdot p_3)\ k^2} = 2\int_0^1\mathrm{d}x\int_0^{1-x}\frac{\mathrm{d}y}{[k^2+2k\cdot(-p_3x+p_2y)]^3}\ .$$

Putting it back into (7), we have

$$\Gamma^\mu(p_2,p_3) = -2\mathrm{i}\left(\frac{4g_\mathrm{s}^2}{3}\right)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathcal{N}^\mu(k,p_2,p_3)}{[k^2+2k\cdot(-p_3x+p_2y)]^3}\ .$$

The $\int\mathrm{d}^n k$ of the six terms of $\mathcal{N}^\mu(k,p_2,p_3)/[k^2+2k\cdot(-p_3x+p_2y)]^3$ – where $\mathcal{N}^\mu(k,p_2,p_3)$ is given in (5) – can be read off directly from the Appendix. For instance, we get for the first term of $\mathcal{N}^\mu(k,p_2,p_3)$

$$\int\mathrm{d}^n k\frac{-4p_2\cdot p_3\gamma^\mu(1-\gamma_5)}{[k^2+2k\cdot(-p_3x+p_2y)]^3} = C\times\left[2(2m^2-q^2)\gamma^\mu(1-\gamma_5)\right]\ ,$$

[1] Marciano, W., Nucl. Phys. **B84** (1975) 132

where C is an overall factor, with $q^2 = (p_2 + p_3)^2 \geq 4m^2$:

$$C \equiv \frac{\mathrm{i}\pi^{\frac{n}{2}}\,\Gamma(3-\frac{n}{2})}{2\,[\mathcal{D}(x,y)]^{3-\frac{n}{2}}}\ ,\quad \mathcal{D}(x,y) = -m^2(x+y)^2 + q^2xy.$$

The result of the integration is now written as follows, with the symbol $\Rightarrow$

$$-4p_2\cdot p_3\gamma^\mu(1-\gamma_5)\ \Rightarrow\ C\times\left[2(2m^2-q^2)\gamma^\mu(1-\gamma_5)\right]\ .$$

In this way, the $\int \mathrm{d}^n k$ integration of the six terms in $\mathcal{N}^\mu(k, p_2, p_3)$ are

$$\begin{aligned}
-4p_2\cdot p_3\gamma^\mu(1-\gamma_5)\ &\Rightarrow\ C\times\left[2(2m^2-q^2)\gamma^\mu(1-\gamma_5)\right]\ ,\\
+4k\cdot(p_2-p_3)\gamma^\mu(1-\gamma_5) &\Rightarrow\ C\times\left[2(q^2-4m^2)(x+y)\gamma^\mu(1-\gamma_5)\right]\ ,\\
+4mk^\mu(1-\gamma_5) &\Rightarrow\ C\times\left[-2m(x+y)(p_2-p_3)^\mu(1-\gamma_5)\right]\ ,\\
-4\,\not{k}(p_2-p_3)^\mu(1-\gamma_5) &\Rightarrow\ C\times\left[+4m(x+y)(p_2-p_3)^\mu\right]\ ,\\
+4m\,\not{k}\gamma^\mu\gamma_5 &\Rightarrow\ C\times\left[-4m^2(x+y)\gamma^\mu\gamma_5 + 8m\,x\,p_3^\mu\gamma_5\right]\ ,\\
(2-n)\,\not{k}\gamma^\mu\,\not{k}(1-\gamma_5) &\Rightarrow \Big\{(2-n)\left[\mathcal{D}(x,y)\gamma^\mu(1-\gamma_5) + (x+y)^2m(p_2-p_3)^\mu\right]\\
&\qquad + \frac{(2-n)^2}{2}\frac{\Gamma(2-\frac{n}{2})}{\Gamma(3-\frac{n}{2})}\mathcal{D}(x,y)\gamma^\mu(1-\gamma_5)\Big\}\times C.
\end{aligned}\tag{14.8}$$

The final result is put into the form

$$\Gamma^\mu(p_2,p_3) = \frac{\frac{4}{3}g_s^2}{(4\pi)^{\frac{n}{2}}}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{\Gamma(3-\frac{n}{2})\,\mathcal{G}^\mu(x,y,p_2,p_3)}{[\mathcal{D}(x,y)]^{3-\frac{n}{2}}}\ , \tag{14.9}$$

where $\mathcal{G}^\mu(x, y, p_2, p_3)$ is the sum of the six terms on the right-hand side of (8), without the overall factor C. The $x \leftrightarrow y$ symmetry of both the denominator $[\mathcal{D}(x, y)]^{3-n/2}$ and the integration domain in (9) implies that the antisymmetric $(x - y)$ terms in (8) vanish after the x, y integrations. These antisymmetric $(x - y)$ terms are therefore not shown in (8). We regroup $\mathcal{G}^\mu(x, y, p_2, p_3)$ into the combination

$$\mathcal{G}^\mu(x,y,p_2,p_3) = a\gamma^\mu + b\frac{(p_2-p_3)^\mu}{2m} - \left[c\gamma^\mu + d\frac{(p_2+p_3)^\mu}{2m}\right]\gamma_5\ , \tag{14.10}$$

$$\begin{aligned}
\text{with}\ \ a &= 2(2m^2-q^2) + 2(q^2-4m^2)(x+y) + (2-n)\mathcal{D}(x,y)\\
&\quad + \frac{(2-n)^2}{2}\frac{\Gamma(2-\frac{n}{2})}{\Gamma(3-\frac{n}{2})}\mathcal{D}(x,y)\ ,\\
b &= 2m^2(x+y)[2+(2-n)(x+y)]\ ,\ \ d = -4m^2(x+y)\ \ ,\ \ c = a-d\ .
\end{aligned}$$

Using the Gordon decomposition, we replace $(p_2 - p_3)^\mu$ with $2m\gamma^\mu - \mathrm{i}\sigma^{\mu\nu}q_\nu$ and group all terms on the right-hand side of (10) into two bases, the vectorial part γ^μ, $\mathrm{i}\sigma^{\mu\nu}q_\nu$ and the axial one $\gamma^\mu\gamma_5$, $q^\mu\gamma_5$. These sets belong to the first-class currents with respect to the G-parity classification, like (12.52). Thus

$$\Gamma^\mu(p_2,p_3) = \gamma^\mu F_1(q^2) + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) - \left[\gamma^\mu G_1(q^2) + \frac{q^\mu}{2m}G_3(q^2)\right]\gamma_5 \,. \tag{14.11}$$

At the tree level, the weak vertex is just a constant $\gamma^\mu(1-\gamma_5)$. When dressed by the gluon to order g_s^2, the one-loop diagram of Fig. 14.4b gives rise to the vector form factors $F_1(q^2)$ and $F_2(q^2)$ as well as to the axial form factors $G_1(q^2)$ and $G_3(q^2)$. The emergence of form factors is intuitively understandable, since surrounded by clouds of gluons and quark pairs, the pointlike quark continuously emits and absorbs virtual particles. It behaves physically as a composite system of virtual particles and hence develops a structure with its own form factors. We have:

$$\begin{aligned}
F_1(q^2) = {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{(4\pi)^{n/2}}\frac{(n-2)^2}{2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\;\frac{\Gamma(2-\frac{n}{2})}{[-m^2(x+y)^2+q^2xy]^{2-\frac{n}{2}}} \\
& + \tfrac{4}{3}\frac{g_\mathrm{s}^2}{8\pi^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\;\frac{m^2[-2+2(x+y)+(x+y)^2]+q^2(1-x)(1-y)}{m^2(x+y)^2-q^2xy}\,, \\
F_2(q^2) = {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{4\pi^2}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\;\frac{m^2(x+y)(1-x-y)}{m^2(x+y)^2-q^2xy}\,, \\
G_1(q^2) = {} & F_1(q^2) - 2F_2(q^2) \quad ; \quad G_3(q^2) = 2F_2(q^2)\,.
\end{aligned} \tag{14.12}$$

In (10) and (12), $\varepsilon = 4-n \neq 0$ is kept only for terms coming along with the singular function $\Gamma(2-\frac{n}{2}) = \Gamma(\varepsilon/2)$. For the regular terms associated with $\Gamma(3-\frac{n}{2})$, the limit $\varepsilon \to 0$ can be safely taken right away. From (3), the typical QCD factor $\frac{4}{3}$ becomes 1 in QED when the gluon in Fig. 14.4b is replaced by a photon. Making the change $u = x+y$, $uv = x-y$, we obtain

$$\begin{aligned}
F_1(q^2) = {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{(4\pi)^{n/2}}\frac{(n-2)^2}{2}\int_0^1\frac{\mathrm{d}u}{u^{3-n}}\int_{-1}^1\left(\frac{\mathrm{d}v}{2}\right)\frac{\Gamma(2-\frac{n}{2})}{\left[-m^2+\frac{1}{4}q^2(1-v^2)\right]^{2-\frac{n}{2}}} \\
& + \tfrac{4}{3}\frac{g_\mathrm{s}^2}{8\pi^2}\int_0^1\frac{\mathrm{d}u}{u}\int_{-1}^1\left(\frac{\mathrm{d}v}{2}\right)\frac{m^2[-2+2u+u^2]+q^2[1-u+\frac{1}{4}u^2(1-v^2)]}{m^2-\frac{1}{4}q^2(1-v^2)}\,, \\
F_2(q^2) = {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{4\pi^2}\int_0^1\mathrm{d}u\int_{-1}^1\left(\frac{\mathrm{d}v}{2}\right)\frac{m^2(1-u)}{m^2-\frac{1}{4}q^2(1-v^2)}\,.
\end{aligned} \tag{14.13}$$

For $F_2(q^2)$, the u,v integrations are straightforward and give

$$\begin{aligned}
F_2(q^2) = {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\frac{1}{\sqrt{\eta(\eta-1)}}\left\{\log\frac{\sqrt{\eta}-\sqrt{\eta-1}}{\sqrt{\eta}+\sqrt{\eta-1}}+\mathrm{i}\,\pi\right\} \quad \text{for } \eta \equiv \frac{q^2}{4m^2} > 1\,, \\
= {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\frac{1}{\sqrt{\eta(\eta-1)}}\log\frac{\sqrt{1-\eta}+\sqrt{-\eta}}{\sqrt{1-\eta}-\sqrt{-\eta}} \quad \text{for } \eta < 0 \quad \xrightarrow[q^2\to 0]{} \tfrac{4}{3}\frac{g_\mathrm{s}^2}{8\pi^2}\,, \\
= {} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{8\pi^2}\frac{1}{\sqrt{\eta(1-\eta)}}\tan^{-1}\sqrt{\frac{\eta}{1-\eta}} \quad \text{for } 0 \le \eta \le 1\,.
\end{aligned} \tag{14.14}$$

This formula is an example showing that form factors are analytic functions in the complex q^2 plane, with cuts on the timelike axis, as mentioned in Chap. 10. By a simple replacement $\frac{4}{3}g_s^2 \to e^2$, we recover the QED corrections to the electron magnetic moment $F_2^{\rm QED}(0)$, first derived by Schwinger,

$$F_2^{\rm QED}(0) = \frac{e^2}{8\pi^2} = \frac{\alpha_{\rm em}}{2\pi} \; . \tag{14.15}$$

The anomalous magnetic moment of the electron, i.e. the deviation from its Bohr magneton value $\mu_e = -e/2m_{\rm e}$, is then $\mu_e(\alpha_{\rm em}/2\pi)$. A pointlike electron with a unit electric form factor and a zero anomalous magnetic moment at the tree level develops its electromagnetic form factors from QED radiative corrections, changing 1 into $1 + F_1^{\rm QED}(q^2)$ and 0 into $F_2^{\rm QED}(q^2)$.

Similarly, we find that QCD generates new weak form factors for quarks. As can be seen from (1) and (11), the modification is

$$1 \to \mathcal{F}_1^{\rm cor}(q^2) = 1 + F_1(q^2) \; , \; 0 \to \left[F_2(q^2) \, , \, G_3(q^2)\right] \; , \; 1 \to 1 + G_1(q^2) \, . \tag{14.16}$$

While quantum corrections make unambiguous finite predictions to $F_2(q^2)$ and $G_3(q^2)$, all the complexities of loop integrals are present in $F_1(q^2)$ [and hence in $G_1(q^2) = F_1(q^2) - 2F_2(q^2)$]. Since the variable of integration k^2 spans the whole range from 0 to ∞, a glance at (4) or (13) reveals two major problems of radiative corrections, both present only in $F_1(q^2)$:

(i) As $k^2 \to \infty$, the integral diverges like d^4k/k^4, and this divergence is transformed into a pole $2/\varepsilon$ via the $\Gamma(2 - \frac{n}{2})$ term contained in the first line of (13). The ultraviolet divergence (UV) has its origin in the *locality* of field theories ($x \to 0$ or equivalently $k \to \infty$); its removal is treated by the *renormalization* program.

(ii) At the lower limit $k^2 \to 0$ of the integration domain, the term k^2 in the denominator of (4) causes a different kind of infinity called infrared divergence (IR). It is identified with the factor $\mathrm{d}u/u$ of $F_1(q^2)$ as $u \to 0$ in (13). Originating from the *massless gluon* via its propagator $1/k^2$, this IR divergence will be canceled by the same IR divergence of the soft gluon bremsstrahlung (Fig. 14.3) that accompanies all radiative processes (Sect. 14.5).

Thus, a study of $F_1(q^2)$ provides an excellent exercise for the understanding of both UV and IR divergences. Looking at the numerator of (7) or at the six terms in (8), we realize that IR and UV divergences come respectively from the constant term $-4p_2 \cdot p_3\gamma^\mu$ and the quadratic term $(2-n) \not{k}\gamma^\mu \not{k}$ in the integration variable k. The four linear terms in k yield finite results. Let us explicitly compute $F_1(q^2)$ in order to single out these two types of divergences.

Consider the first part of $F_1(q^2)$ in the first line of (13), i.e. the part that contains the singular $\Gamma(2 - \frac{n}{2})$ factor, we call it $F_{1,\rm uv}(q^2)$. Besides $\Gamma(2 - \frac{n}{2}) = \frac{2}{\varepsilon} - \gamma_E + O(\varepsilon)$, where $\gamma_E \approx 0.5772$ is the Euler constant, we also expand all other terms of $F_{1,\mathrm{uv}}(q^2)$ up to order ε such that they can cancel the singularity $2/\varepsilon$ when multiplied by the latter. The result is

$$\begin{aligned} F_{1,\mathrm{uv}}(q^2) =& \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\left\{-\int_0^1 \mathrm{d}v\ \log\left[1-\frac{q^2}{4m^2}(1-v^2)\right]\right. \\ & \left.+\frac{2}{\varepsilon}-\gamma_E+\log(4\pi)-1-\log\frac{m^2}{\mu^2}\right\}\ , \\ \xrightarrow[q^2\to 0]{} & \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\left[\frac{2}{\varepsilon}-\gamma_E+\log(4\pi)-1-\log\frac{m^2}{\mu^2}\right]\ . \end{aligned} \tag{14.17}$$

On dimensional grounds, a mass scale factor μ must enter the denominator $\Delta^{2-\frac{n}{2}} \equiv [-m^2+q^2(1-v^2)/4]^{2-\frac{n}{2}}$ of (13) in order to make the latter dimensionally correct for $n \neq 4$. Indeed, for small $\varepsilon \neq 0$, one can always write the term $\Delta^{\frac{\varepsilon}{2}}$ as $1+\frac{\varepsilon}{2}\ \log(\Delta/\mu^2)+\mathcal{O}(\varepsilon^2)$. This arbitrary factor μ^2 serves as a reminder that the decomposition of a divergent integral into an infinite term and a finite term always involves an ambiguity. Only a coherent procedure that removes the ε pole and gives finite values *independent of* μ^2 to physical quantities is meaningful. This is the essence of the renormalization.

The second part of $F_1(q^2)$ on the second line of (13) – the part without the singular $\Gamma(2-\frac{n}{2})$ – is either finite or infrared divergent. The factor $\mathrm{d}u/u$ is the source of IR divergence when $u\to 0$. To neutralize $1/u$, we group in the numerator of the integrand all the terms which are linear and quadratic in the variable u. They are $m^2(2u+u^2)+q^2(-u+u^2(1-v^2)/4)$. These terms cancel $1/u$ and yield a finite result denoted by $F_{1,\mathrm{fi}}(q^2)$ after the u, v integrations:

$$\begin{aligned} F_{1,\mathrm{fi}}(q^2) =& \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\left\{-1\ +\frac{3-4\eta}{\sqrt{\eta(\eta-1)}}\left[\log\frac{\sqrt{\eta}-\sqrt{\eta-1}}{\sqrt{\eta}+\sqrt{\eta-1}}+\mathrm{i}\pi\right]\right\}\ , \\ \xrightarrow[q^2\to 0]{} & \tfrac{4}{3}\frac{5g_\mathrm{s}^2}{16\pi^2}\ . \end{aligned} \tag{14.18}$$

The remainder $(-2m^2+q^2)$, which cannot cancel $1/u$, will give rise to an infrared divergent term, $F_{1,\mathrm{ir}}(q^2)$. We have

$$F_{1,\mathrm{ir}}(q^2) = \tfrac{4}{3}\frac{g_\mathrm{s}^2}{8\pi^2}\int_0^1 \mathrm{d}v\frac{-2m^2+q^2}{m^2-\frac{1}{4}q^2(1-v^2)}\int_0^1\frac{\mathrm{d}u}{u} \xrightarrow[q^2\to 0]{} -\tfrac{4}{3}\frac{g_\mathrm{s}^2}{4\pi^2}\int_0^1\frac{\mathrm{d}u}{u}. \tag{14.19}$$

The sum of the three contributions (17), (18), (19) at the limit $q^2=0$ gives

$$F_1(0) = \tfrac{4}{3}\frac{g_\mathrm{s}^2}{16\pi^2}\left\{\frac{2}{\varepsilon}-\gamma_E+\log(4\pi)-\log\left(\frac{m^2}{\mu^2}\right)+4-4\int_0^1\frac{\mathrm{d}u}{u}\right\}. \tag{14.20}$$

The UV and IR divergences are represented by $2/\varepsilon$ and $\int_0^1 \mathrm{d}u/u$ respectively. The reason for writing the explicit expression of $F_1(0)$ will be clear later. We have seen that $\Gamma^\mu(p_2,p_3)$, hence $F_1(q^2)$, has two types of divergence. Let us concentrate first on UV, leaving the treatment of IR to Sect. 14.4.

14.2 Quark Self-Energy

To order g_s^2 corrections to the tree vertex $\gamma^\mu(1-\gamma_5)$, besides the diagram of Fig. 14.2a that we have just considered, there are two more diagrams shown in Fig. 14.2b, c which also contribute. They are related to the quark self-energy $\Sigma(p)$, in which the generic p stands for the external momenta.
The expression for the self-energy of Fig. 14.5 can be written as

$$-\mathrm{i}\Sigma(p) \stackrel{\text{def}}{=} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(-\mathrm{i}g_s\gamma^\rho\frac{\lambda_j}{2}\right)\frac{\mathrm{i}}{\not p-\not k-m+\mathrm{i}\eta}\left(-\mathrm{i}g_s\gamma^\sigma\frac{\lambda_j}{2}\right)\frac{-\mathrm{i}g_{\rho\sigma}}{k^2+\mathrm{i}\eta}. \quad (14.21)$$

Like the vertex function $\Gamma^\mu(p_2,p_3)$ discussed in (4), we can at the outset use the Feynman–'t Hooft gauge $\xi=1$ for the gluon propagator in (21). In the definition of $-\mathrm{i}\Sigma(p)$, we note that an additional factor $(-\mathrm{i})$ is included, its convenience is clearly seen in (22). Apart from a common factor, the three amplitudes of Fig. 14.2 can be written respectively from left to right as

$$\Gamma^\mu(p_2,p_3)\ ,\quad \frac{\mathrm{i}\left[-\mathrm{i}\Sigma(p_2)\right]}{\not p_2-m}\gamma^\mu=\frac{\Sigma(p_2)}{\not p_2-m}\gamma^\mu\ ,\quad \text{and}\quad \gamma^\mu\frac{\Sigma(p_3)}{\not p_3-m}\ . \quad (14.22)$$

A glance at the integral in (21) shows that both UV and IR divergences are present in $\Sigma(p)$. After writing the denominator of (21) as an integral of the Feynman auxiliary variable x and integrating over k, we obtain

$$\begin{aligned}
\Sigma(p) &= -\left(\tfrac{4}{3}\right)\mathrm{i}g_s^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{n\,m-(n-2)(\not p-\not k)}{[k^2-2p.kx+(p^2-m^2)x]^2}\ ,\\
&\equiv mA(p^2)+\not pB(p^2)\ ,\\
A(p^2) &= \tfrac{4}{3}\frac{g_s^2}{(4\pi)^{n/2}}\int_0^1\mathrm{d}x\ \frac{n\Gamma(2-\frac{n}{2})}{[p^2x(1-x)-m^2x]^{2-\frac{n}{2}}}=\tfrac{4}{3}\frac{g_s^2}{4\pi^2}\left(\frac{2}{\varepsilon}+\cdot\cdot\right),\\
B(p^2) &= \tfrac{4}{3}\frac{g_s^2}{(4\pi)^{n/2}}\int_0^1\mathrm{d}x\ \frac{(2-n)\Gamma(2-\frac{n}{2})(1-x)}{[p^2x(1-x)-m^2x]^{2-\frac{n}{2}}}=\tfrac{4}{3}\frac{-g_s^2}{16\pi^2}\left(\frac{2}{\varepsilon}+\cdot\cdot\right).
\end{aligned} \quad (14.23)$$

Besides their UV divergences with the ε pole, the dimensionless quantities $A(p^2)$, $B(p^2)$ also have their IR divergences coming from $x\to 0$.

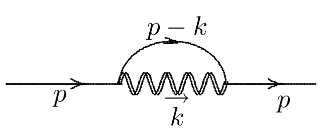

Fig. 14.5. Fermionic self-energy $\Sigma(p)$

The removal of the ultraviolet divergences in $\Gamma^\mu(p_2,p_3)$ and $\Sigma(p)$ is at the heart of the renormalization program which can be stated as follows: The ultraviolet infinities encountered in $F_1(q^2)$ and $\Sigma(p)$ can be consistently removed and the final results become finite if they are expressed in terms of the *renormalized quark mass* together with the *renormalized quark field*. We now go further by introducing the notion of *bare* and *renormalized* quantities.

14.3 Mass and Field Renormalization

Let us recall the relation between the propagator and the mass of a particle. The latter is associated with the pole of the former, so the mass is the solution of the equation obtained by setting to zero the inverse of its propagator. How can the mass of a fermion be determined if viewed as a cloud of virtual particles that are continuously being created and destroyed? Starting from a bare mass m_0, the fermion develops a change in m_0 by interacting with gluons through loop diagrams, the simplest example $\Sigma(p)$ is shown in Fig. 14.5. $\Sigma(p)$ is called self-energy, i.e. the extra amount of rest mass energy *generated by quantum corrections*. For the moment this change is infinite. If the mass is changed, so is the propagator. The bare fermion propagator is then replaced by its *full* or *dressed* propagator which includes all possible loop diagrams having two external fermion lines, also called the two-point Green's function.

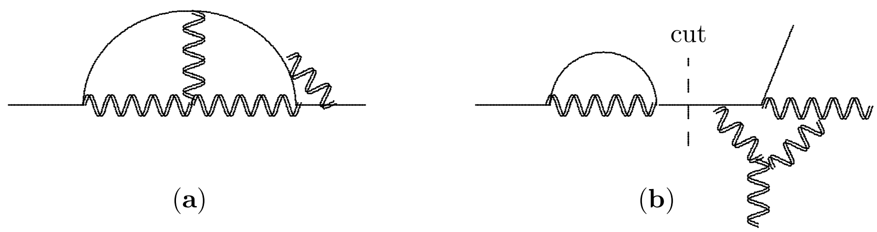

Fig. 14.6. (**a**) Irreducible $1PI$ diagram; (**b**) reducible $1PR$ diagram

To compute the dressed propagator, let us define a *one-particle irreducible* $1PI$ diagram to be any loop diagram that cannot be split into two disconnected ones by cutting only a single internal line. Every diagram is either irreducible ($1PI$) or reducible ($1PR$). An example of $1PI$ is Fig. 14.6a while Fig. 14.6b illustrates a $1PR$ diagram. The reason to select $1PI$ is that any $1PR$ can be decomposed into a set of $1PI$ without further loops, such that, if we know how to handle the UV divergences in $1PI$, then these UV divergences are also under control in $1PR$.

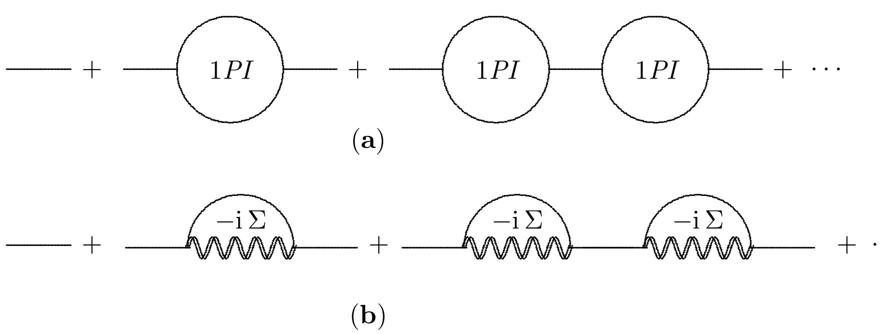

Fig. 14.7. (**a**) Geometric sum of $1PI$ diagrams; (**b**) dressed propagator $S_{\mathrm{D}}(p)$

Dressed Propagator. A dressed propagator is an infinite geometric sum of $1PI$ diagrams with two external lines (Fig. 14.7a). Each $1PI$ by itself is an infinite sum of diagrams of orders $g_\mathrm{s}^2, g_\mathrm{s}^4 \cdots$, the lowest $1PI$ starting at g_s^2 is the diagram of Fig. 14.5 denoted by $-\mathrm{i}\Sigma(p)$ in (21).

To this order g_s^2 shown in Fig. 14.7b, the dressed quark propagator $S_\mathrm{D}(p)$ is obtained by summing up the geometric series of $-\mathrm{i}\Sigma(p)$. We have

$$\begin{aligned} S_\mathrm{D}(p) &= \frac{\mathrm{i}}{\not p - m_0} + \frac{\mathrm{i}}{\not p - m_0}[-\mathrm{i}\Sigma(p)]\frac{\mathrm{i}}{\not p - m_0} + \\ &\quad + \frac{\mathrm{i}}{\not p - m_0}[-\mathrm{i}\Sigma(p)]\frac{\mathrm{i}}{\not p - m_0}[-\mathrm{i}\Sigma(p)]\frac{\mathrm{i}}{\not p - m_0} + \cdots \\ &= \frac{\mathrm{i}}{\not p - m_0}\left[1 + \frac{\Sigma(p)}{\not p - m_0} + \left(\frac{\Sigma(p)}{\not p - m_0}\right)^2 + \cdots\right] \\ &= \frac{\mathrm{i}}{\not p - m_0}\left(1 - \frac{\Sigma(p)}{\not p - m_0}\right)^{-1} = \frac{\mathrm{i}}{\not p - m_0 - \Sigma(p)}\ . \end{aligned} \tag{14.24}$$

With quantum corrections, the physical mass m is now associated with the pole of the dressed propagator $S_\mathrm{D}(p)$ in (24). The pole is no longer the bare mass m_0 but is shifted to m, solution to the equation

$$\begin{aligned} &\not p - m_0 - \Sigma(\not p = m) = 0\ , \qquad \text{or} \\ &m = m_0 + \Sigma(\not p = m) = m_0 + m_0 A(m^2) + m B(m^2)\ . \end{aligned} \tag{14.25}$$

It is important to remark that in the above equation, $\Sigma(p)$ can be considered as a function of $\not p$, for $p^2 = (\not p)^2$. For instance $\Sigma(m)$ is understood as $\Sigma(\not p = m)$. In the expressions of $A(p^2)$ and $B(p^2)$ given by (23), m must be understood as m_0, since $\Sigma(p)$ was computed with the bare mass m_0. For example the $B(m^2)$ in (25) must be read as

$$B(m^2) = \frac{4}{3}\frac{(2-n)\ g_\mathrm{s}^2}{(4\pi)^{n/2}}\Gamma\left(2 - \tfrac{n}{2}\right)\int_0^1 \mathrm{d}x\ \frac{(1-x)}{[m^2 x(1-x) - m_0^2 x]^{2-\frac{n}{2}}}\ . \tag{14.26}$$

Close to the physical pole $\not p \approx m$, we write the Taylor expansion

$$\Sigma(p) = \Sigma(m) + (\not p - m)\ \frac{\mathrm{d}\Sigma(p)}{\mathrm{d}\ \not p}\bigg|_{\not p = m} + \mathcal{O}(p^2 - m^2)\ . \tag{14.27}$$

With $\Sigma(m) = m - m_0$, the denominator of $S_\mathrm{D}(p)$ in (24) has the following form for $\not p \approx m$:

$$\begin{aligned} \not p - m_0 - \Sigma(p)\ &\approx \not p - m_0 - \Sigma(m) - (\not p - m)\ \frac{\mathrm{d}\Sigma(p)}{\mathrm{d}\ \not p}\bigg|_{\not p = m}\ , \\ &\approx\ (\not p - m)\left(1 - \frac{\mathrm{d}\Sigma(p)}{\mathrm{d}\ \not p}\bigg|_{\not p = m}\right)\ . \end{aligned} \tag{14.28}$$

The shift of the mass $m_0 \to m = m_0 + \Sigma(m)$ not only brings a new pole at $\not{p} = m$ to the dressed propagator $S_{\rm D}(p)$ but also changes its residue at the physical pole m:

$$\begin{array}{ccc} \text{Bare propagator} & & \text{Dressed propagator} \\ \dfrac{\rm i}{\not{p} - m_0} & \longrightarrow & S_{\rm D}(p) = \dfrac{{\rm i}\, Z_{\rm q}}{\not{p} - m}\ , \end{array} \tag{14.29}$$

where $Z_{\rm q}$ is given by

$$\frac{1}{Z_{\rm q}} = 1 - \left.\frac{{\rm d}\Sigma(p)}{{\rm d}\not{p}}\right|_{\not{p}=m} . \tag{14.30}$$

The above expression for $Z_{\rm q}$ is derived by comparing (28) with the inverse of the dressed propagator given in (24). Indeed

$$S_{\rm D}^{-1}(p) = \frac{1}{\rm i}(\not{p} - m_0 - \Sigma(p)) = \frac{1}{\rm i}(\not{p} - m)\left(1 - \left.\frac{{\rm d}\Sigma(p)}{{\rm d}\not{p}}\right|_{\not{p}=m}\right) \equiv \frac{\not{p} - m}{{\rm i}\, Z_{\rm q}} .$$

Equation (30), which relates the residue $Z_{\rm q}$ to the derivative of the self-energy $\Sigma(p)$ at the physical mass $\not{p} = m$, is in principle formally valid when $\Sigma(p)$ and its derivative are finite. However, if they are infinite, (30) might be misleading since it would give $Z_{\rm q} \to 0$ for ${\rm d}\Sigma(p)/{\rm d}\not{p}|_{\not{p}=m} \to \infty$ which is absurd. To understand this subtlety, let us remark that without interactions, i.e. at zero order of the coupling constant $g_{\rm s}$, $Z_{\rm q} \equiv 1$, $\Sigma(p) \equiv 0$, which is of course satisfied by (30). The perturbative quantum corrections start at order $g_{\rm s}^2$: $\Sigma(p) = \mathcal{O}(g_{\rm s}^2)$, $Z_{\rm q} = 1 + \mathcal{O}(g_{\rm s}^2)$. Within the framework of perturbative calculations in which $Z_{\rm q}$ is derived, (30) should be correctly written as

$$\left.\frac{{\rm d}\Sigma(p)}{{\rm d}\not{p}}\right|_{\not{p}=m} = 1 - \frac{1}{Z_{\rm q}} = \frac{Z_{\rm q} - 1}{Z_{\rm q}} = Z_{\rm q} - 1 + \mathcal{O}(g_{\rm s}^4) . \tag{14.31}$$

The above relation satisfies the $\mathcal{O}(g_{\rm s}^2)$ expansion. In the spirit of perturbative calculations, one has

$$Z_{\rm q} = 1 + \left.\frac{{\rm d}\Sigma(p)}{{\rm d}\not{p}}\right|_{\not{p}=m} , \quad \sqrt{Z_{\rm q}} = 1 + \frac{1}{2}\left.\frac{{\rm d}\Sigma(p)}{{\rm d}\not{p}}\right|_{\not{p}=m} . \tag{14.32}$$

Let us recall that the propagator is the Fourier transform of a two-point function. The bare propagator is associated with $\langle 0 \,|\, T[\psi_0(x)\overline{\psi}_0(y)] \,|\, 0\rangle$ of the bare field $\psi_0(x)$, the dressed propagator $S_D(p)$ is computed from the bare field $\psi_0(x)$ too [by summation over the geometric series in (24)].

This two-point function definition of the propagator suggests that if we scale the bare field ψ_0 by $1/\sqrt{Z_{\rm q}}$ and define the *renormalized* field ψ by $\psi_0(x) = \sqrt{Z_{\rm q}}\psi(x)$, the infinite residue $Z_{\rm q}$ of $S_{\rm D}(p)$ can be absorbed by ψ_0, and $S_{\mathrm{D}}(p)$ is promoted to the *renormalized propagator* $\widetilde{S}_{\mathrm{ren}}(p)$ with *pole* at $\not{p} = m$ and with *residue* $= 1$. Thus,

$$S_{\mathrm{D}}(p) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}p\cdot x}\ \left\langle 0 \,\middle|\, T(\psi_0(x)\overline{\psi}_0(0)) \,\middle|\, 0 \right\rangle = \frac{\mathrm{i}\, Z_{\mathrm{q}}}{\not{p} - m}\ .$$

Putting $\psi_0(x) = \sqrt{Z_{\mathrm{q}}}\psi(x)$, one has

$$\widetilde{S}_{\mathrm{ren}}(p) = \int \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}p\cdot x}\ \left\langle 0 \,\middle|\, T(\psi(x)\overline{\psi}(0)) \,\middle|\, 0 \right\rangle = \frac{\mathrm{i}}{\not{p} - m}\ . \tag{14.33}$$

Let us clarify again the meaning of the shift of the fields proposed in (33). We have two quantities m and Z_{q} governed by two equations (25) and (31). If they can be solved, we would obtain physical quantities m and Z_{q} and could stop here. It must be emphasized that the renormalization of masses and fields has nothing directly to do with infinities encountered in the computation of the self-energy $\Sigma(p)$. It would still be necessary even in a theory in which loop integrals were convergent. However, since $\Sigma(p)$ is infinite, m cannot be computed but is only fixed by the physical mass. The bare mass parameter m_0 (which is infinite) is adjusted to $\Sigma(\not{p} = m)$ in (25) to cancel its UV divergences and giving the physical finite mass m. Similarly, the infinite Z_{q} is adjusted to bare field ψ_0 (which is also infinite) to define the renormalized field $\psi = \psi_0/\sqrt{Z_{\mathrm{q}}}$, such that the propagator of a renormalized field has a pole at the physical mass and has a residue equal to unit, thus

$$\begin{array}{ccccccc} \text{Bare} & & \text{Interaction} & & \text{Dressed} & & \text{Renormalized} \\ \dfrac{\mathrm{i}}{\not{p} - m_0} & \longrightarrow & \Sigma(p) & \longrightarrow & \dfrac{\mathrm{i}\, Z_{\mathrm{q}}}{\not{p} - m} & \longrightarrow & \dfrac{\mathrm{i}}{\not{p} - m}\ . \end{array} \tag{14.34}$$

The procedure by which the UV divergence in Z_{q} is removed from the theory is referred to as the *field strength renormalization* or the *wave function renormalization*, likewise, the removal of the UV divergence in $\Sigma(p)$ is referred to as the *mass renormalization*. The role of Z_{q} (and other Z_j introduced later) is essential here and in the next chapters.

The concept of removal of the infinities is not restricted to quantum field theory. Even in classical electrodynamics, the self-energy of a pointlike electron which interacts with its own electric field is also divergent. The idea of subtraction of infinities – the core of the renormalization concept – was first suggested by Kramers. He observed that although the self-energy of a pointlike electron is infinite, actually the meaningful quantity is the *difference* between the self-energy of the free electron and that of the electron bound in an atom. Both of these self-energies diverge, but their difference is finite.

Counterterms. A counterterm is formally an infinite parameter introduced in the Lagrangian to remove (or absorb) the UV divergences by adjusting the bare parameters. After a subtraction of the infinities, the finite parts (also called the renormalized quantities) are constrained to obey *renormalization conditions* (also called normalization or subtraction conditions or prescriptions). In the case discussed here, the renormalization conditions dictate that the residue of the renormalized propagator is 1, and $\Sigma(m)+m_0 = m$. The number of counterterms must be limited, independently of the perturbative orders, otherwise the theory is nonrenormalizable.

For this renormalization program to work, it is essential that the original Lagrangian includes all interactions generated by the UV parts of Feynman amplitudes. This is the case of the standard electroweak theory and QCD. If for some reason, the loop integrals produce a UV term which has a covariant structure that the original Lagrangian does not possess, this UV divergence cannot be removed. For instance, if the magnetic moment $F_2(q^2)$ found in (14) were infinite, there is no way to absorb its divergence into the original QCD or QED Lagrangian since the latter does not possess such a Pauli magnetic interaction $(g_\mathrm{s}/m)\,[\partial_\mu A^k_\nu(x) - \partial_\nu A^k_\mu(x)]\,\overline{\psi}(x)\lambda_k\sigma^{\mu\nu}\psi(x)$ from the start. The fact that $F_2(q^2)$ is finite is not an accident, it reveals the renormalizability of QCD. The renormalizability of QCD and QED forbids the presence of the Pauli magnetic interaction although Lorentz and gauge invariances allow such a term. The magnetic interaction will provoke an avalanche of infinities in loop integrals since the dimensional coupling constant g_s/m implies a momentum dependence of the vertex. In turn it makes the loop integrals more and more divergent with increasing perturbative orders and an infinite numbers of counterterms are needed. This interaction is nonrenormalizable.

14.3.1 Renormalized Form Factor $\widetilde{F}_1^{\mathrm{ren}}(q^2)$

Having outlined the concept of renormalization, we now apply the method to our problem of removing the ultraviolet divergences in $F_1(q^2)$ and $\Sigma(p)$. This procedure promotes $F_1(q^2)$ to the renormalized form factor $\widetilde{F}_1^{\mathrm{ren}}(q^2)$, and $\Sigma(p)$ to the renormalized self-energy $\widetilde{\Sigma}_{\mathrm{ren}}(p)$; both of them are free of UV divergences. Let us see how it works.

In the Lagrangian $\mathcal{L}$, the bare quark fields $\psi^0(x)$ coupled with the boson field $W_\mu(x)$ can be written as (the weak coupling constant g_W is implicit)

$$\mathcal{L} = \overline{\psi}^0_{\mathrm{q}_2}(x)\gamma^\mu(1-\gamma_5)\psi^0_{\mathrm{q}_3}(x)W_\mu(x) + \ \text{kinetic term}\ , \tag{14.35}$$

where the kinetic term $\overline{\psi^0}(x)(\mathrm{i}\,\partial\!\!\!/ - m_0)\psi^0(x)$ is applied to both quark fields $\psi^0_{\mathrm{q}_2}(x)$ and $\psi^0_{\mathrm{q}_3}(x)$. We define the renormalized fields $\psi_{\mathrm{q}_2}(x)$ and $\psi_{\mathrm{q}_3}(x)$ by introducing the counterterms $\sqrt{Z_{\mathrm{q}_2}}$ and $\sqrt{Z_{\mathrm{q}_3}}$. The W boson field $W_\mu(x)$, not affected by gluons, is untouched. We write

$$\psi^0_{\mathrm{q}_\mathrm{j}}(x) = \sqrt{Z_{\mathrm{q}_\mathrm{j}}}\,\psi_{\mathrm{q}_\mathrm{j}}(x)\ \ ,\ \ \mathrm{j}\ = 2,3\ . \tag{14.36}$$

Putting $\psi^0_{\mathrm{q}_\mathrm{j}}(x) = \sqrt{Z_{\mathrm{q}_\mathrm{j}}}\,\psi_{\mathrm{q}_\mathrm{j}}(x)$ back into (35), the original $\mathcal{L}$ is now split into two parts:

$$\begin{aligned}
&\mathcal{L} = \mathcal{L}_{\mathrm{ren}} + \mathcal{L}_{\mathrm{ct}}\ ;\\
&\mathcal{L}_{\mathrm{ren}} = \overline{\psi}_{\mathrm{q}_2}(x)\gamma^\mu(1-\gamma_5)\psi_{\mathrm{q}_3}(x)W_\mu(x)\ ,\\
&\mathcal{L}_{\mathrm{ct}} = \delta_\mathrm{q}\,\overline{\psi}_{\mathrm{q}_2}(x)\gamma^\mu(1-\gamma_5)\psi_{\mathrm{q}_3}(x)W_\mu(x)\ \ ;\ \ \delta_\mathrm{q} = \sqrt{Z_{\mathrm{q}_2}}\sqrt{Z_{\mathrm{q}_3}} - 1\ ,
\end{aligned} \tag{14.37}$$

the kinetic term of (35), which should appear in the above equation, will be given later in (44). The role of $\mathcal{L}_{\rm ct}$ is to cancel the UV divergences of $\mathcal{L}_{\rm ren}$. Physical quantities, which are obtained by summing the contributions from both $\mathcal{L}_{\rm ren}$ and $\mathcal{L}_{\rm ct}$, are free of UV divergences.

To first order of $\delta_{\rm q}$, $\mathcal{L}_{\rm ct}$ gives a contribution $\delta_{\rm q}\,\gamma^\mu(1-\gamma_5)$ to the vertex $\gamma^\mu(1-\gamma_5)$. Note that $\delta_{\rm q}$ is $\mathcal{O}(g_{\rm s}^2)$. To the same $g_{\rm s}^2$ order, $\mathcal{L}_{\rm ren}$ gives $\Gamma^\mu(p_2,p_3)$ in (12) for which m is now understood as the renormalized mass. The sum of $\delta_{\rm q}\,\gamma^\mu(1-\gamma_5)$ with $\Gamma^\mu(p_2,p_3)$ is the ultraviolet-convergent renormalized vertex function $\widetilde{\Gamma}^\mu_{\rm ren}(p_2,p_3)$ which replaces $\Gamma^\mu(p_2,p_3)$:

$$\Gamma^\mu(p_2,p_3) \longrightarrow \widetilde{\Gamma}^\mu_{\rm ren}(p_2,p_3) = \Gamma^\mu(p_2,p_3) + \delta_{\rm q}\;\gamma^\mu(1-\gamma_5)\,. \tag{14.38}$$

Comparing (38) with (11), we get the renormalized form factor $\widetilde{F}_1^{\rm ren}(q^2)$ which replaces $F_1(q^2)$:

$$F_1(q^2) \longrightarrow \widetilde{F}_1^{\rm ren}(q^2) = F_1(q^2) + \delta_{\rm q}\,. \tag{14.39}$$

Our next task is to compute $\delta_{\rm q}$. From (32) and (37), we have

$$\delta_{\rm q} = \frac{1}{2}\left\{ \frac{{\rm d}\Sigma(p_2)}{{\rm d}\,\not p_2}\bigg|_{\not p_2=m_2} + \frac{{\rm d}\Sigma(p_3)}{{\rm d}\,\not p_3}\bigg|_{\not p_3=m_3}\right\} = \frac{{\rm d}\Sigma(p)}{{\rm d}\,\not p}\bigg|_{\not p=m}\,, \tag{14.40}$$

since we take $m_2=m_3=m$. The derivative of $\Sigma(p)$ can be readily performed using $\Sigma(p)$ given by (23). We obtain

$$\begin{aligned}\delta_{\rm q} = \frac{{\rm d}\Sigma(p)}{{\rm d}\,\not p}\bigg|_{\not p=m} &= B(m^2) + 2m^2\,\frac{{\rm d}A(p^2)}{{\rm d}p^2}\bigg|_{p^2=m^2} + 2m^2\,\frac{{\rm d}B(p^2)}{{\rm d}p^2}\bigg|_{p^2=m^2}\\ &= \tfrac{4}{3}\frac{g_{\rm s}^2}{(4\pi)^{\frac{n}{2}}}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \frac{{\rm d}x(1-x)}{[-m^2x^2]^{2-\frac{n}{2}}}\left\{-2+3\varepsilon+\frac{2\varepsilon}{x}\right\}\,.\end{aligned}$$

Developing all the terms of the above equation in a series of ε, we get

$$\delta_{\rm q} = \tfrac{4}{3}\frac{g_{\rm s}^2}{16\pi^2}\left[\frac{-2}{\varepsilon} + \gamma_E - \log(4\pi) + \log\left(\frac{m^2}{\mu^2}\right) - 4 + 4\int_0^1 \frac{{\rm d}x}{x}\right]\,. \tag{14.41}$$

Comparing (41) with $F_1(0)$ as given by (20), we get a remarkable result; to wit, the quantity $\delta_{\rm q}$ *exactly cancels* $F_1(0)$:

$$\delta_{\rm q} + F_1(0) = 0\;\;,\;\;\text{or}\quad \frac{{\rm d}\Sigma(p)}{{\rm d}\,\not p}\bigg|_{\not p=m} + F_1(0) = 0\,. \tag{14.42}$$

This important result, due to the current conservation, in fact mimics the Ward identity in QED (Problem 14.3) according to which the apparently unrelated quantities $F_1(0)$ and ${\rm d}\Sigma(p)/{\rm d}\,\not p|_{\not p=m}$ turn out to satisfy (42). In the case considered here, since we take $m_2 = m_3 = m$, the weak vector current is conserved, hence we also obtain (42). With (39), we have

$$\widetilde{F}_1^{\rm ren}(q^2) = F_1(q^2) - F_1(0) \ . \tag{14.43}$$

The expression (43) is pleasantly simple. Both $F_1(q^2)$ and $F_1(0)$ are separately divergent, but their difference $\widetilde{F}_1^{\rm ren}(q^2)$ is finite (free of UV divergences more precisely). The $2/\varepsilon$ pole of $F_{1,\rm uv}(q^2)$ in (17) cancels the same $2/\varepsilon$ pole of $F_1(0)$. The counterterm used to absorb the ultraviolet divergence in $F_1(q^2)$ is precisely the value of $F_1(q^2)$ itself at $q^2 = 0$.

From now on, $F_1(q^2)$ is replaced with $\widetilde{F}_1^{\rm ren}(q^2) = F_1(q^2) - F_1(0)$. Notice that the equation $\widetilde{F}_1^{\rm ren}(0) = 0$ represents the *renormalization condition.* When we look back at $\mathcal{F}_1^{\rm cor}(q^2)|_{\rm ren} = 1 + \widetilde{F}_1^{\rm ren}(q^2)$ in (16), we realize that at $q^2 = 0$, QCD does not bring any modification to $\mathcal{F}_1^{\rm cor}(0)|_{\rm ren}$.

14.3.2 Important Consequence of Mass Renormalization

To renormalize the quark mass, we need a counterterm Δm_0 to be added to the bare mass m_0 to get the physical mass m, $\Delta m_0 + m_0 = m$. Together with $Z_{\rm q}$, this Δm_0 brings in $\mathcal{L}_{\rm ct}$ a new term to be added to $\Sigma(p)$. Their sum is the renormalized self-energy $\widetilde{\Sigma}_{\rm ren}(p)$, which is found to be

$$\widetilde{\Sigma}_{\rm ren}(p) = \Sigma(p) - (Z_{\rm q} - 1)(\not{p} - m) - Z_{\rm q}\, \Delta m_0 \ . \tag{14.44}$$

The generic p stands for p_2 or p_3 and m stands for m_2 or m_3. Like $\delta_{\rm q}$ in (37), the last two terms of (44) arise when we put $\psi^0 = \sqrt{Z_{\rm q}}\,\psi$ in the kinetic expression $\overline{\psi^0}({\rm i}\,\not{\partial} - m_0)\psi^0$ of $\mathcal{L}$ in (35). Like (38) where $\Gamma^\mu(p_2, p_3)$ is replaced with $\widetilde{\Gamma}^\mu_{\rm ren}(p_2, p_3)$, now $\Sigma(p)$ is replaced with $\widetilde{\Sigma}_{\rm ren}(p)$. Thus, the amplitudes of Fig. 14.2b, c are now expressed in terms of $\widetilde{\Sigma}_{\rm ren}(p)$. From (22), they are given by

$$\frac{\widetilde{\Sigma}_{\rm ren}(p_2)}{\not{p}_2 - m_2}\gamma^\mu \quad \text{and} \quad \gamma^\mu \frac{\widetilde{\Sigma}_{\rm ren}(p_3)}{\not{p}_3 - m_3} \ . \tag{14.45}$$

Now comes an important result which eliminates the contributions of both Fig. 14.2b, c. The renormalization condition, which sets the pole of the renormalized propagator $\widetilde{S}_{\rm ren}(p)$ to be m, implies that $\widetilde{\Sigma}_{\rm ren}(\not{p} = m) = 0$. This equation is obtained from (25) and (44). Indeed, keeping in mind that $Z_{\rm q} = 1 + \mathcal{O}(g_{\rm s}^2)$ and Δm_0 is also $\mathcal{O}(g_{\rm s}^2)$, one has

$$\widetilde{\Sigma}_{\rm ren}(\not{p} = m) = \Sigma(\not{p} = m) - Z_{\rm q}\Delta m_0 = \Sigma(\not{p} = m) - \Delta m_0 = 0 \ . \tag{14.46}$$

Not only is $\widetilde{\Sigma}_{\rm ren}(\not{p} = m)$ equal to zero, its derivative at $\not{p} = m$ also vanishes, using (44) and (40) and remembering that $\delta_{\rm q} = Z_{\rm q} - 1$. We therefore have

$$\widetilde{\Sigma}_{\rm ren}(\not{p} = m) = 0 \quad \text{and} \quad \left.\frac{{\rm d}\widetilde{\Sigma}_{\rm ren}(\not{p})}{{\rm d}\,\not{p}}\right|_{\not{p}=m} = 0 \ . \tag{14.47}$$

Equation (47) has an important consequence which forces (45) to vanish. The demonstration is straightforward using the Taylor expansion similar to (27). Indeed, when

$$\widetilde{\Sigma}_{\mathrm{ren}}(\not p = m) = 0 \quad \text{and} \quad \frac{\mathrm{d}\,\widetilde{\Sigma}_{\mathrm{ren}}(\not p)}{\mathrm{d}\,\not p}\bigg|_{\not p = m} = 0\ ,\ \text{then}\ \frac{\widetilde{\Sigma}_{\mathrm{ren}}(m)}{\not p - m} = 0\ . \tag{14.48}$$

From (45) and (48), we note that after the quark gets its renormalized mass, the self-energy amplitude of each of the two diagrams (Fig. 14.2b, c) vanishes. This important property, established for fermionic fields, can be generalized to boson fields too. We arrive at the essential point: *for external particles (fermion or boson) on the mass-shell, it is not necessary to include self-energy corrections*. Via the self-energy $\Sigma(p)$, the two diagrams in Fig. 14.2b–c have no other role than to provide indirectly the counterterm δ_{q} to the vertex $\gamma^\mu(1-\gamma_5)$ in (38), then cease to contribute.

The detailed studies in the three previous sections can be summarized as follows. The ultraviolet divergences of the three diagrams in Fig. 14.2 are absorbed by the renormalization procedure into the bare parameters (fields and masses) of the Lagrangian. It results in replacing $\Gamma^\mu(p_2,p_3)$ with the renormalized $\widetilde{\Gamma}^\mu_{\mathrm{ren}}(p_2,p_3)$, leading to the key result $F_1(q^2) \longrightarrow \widetilde{F}_1^{\mathrm{ren}}(q^2) = F_1(q^2) - F_1(0)$ which is ultraviolet convergent.

14.4 Virtual Gluon Contributions

We have determined $\widetilde{F}_1^{\mathrm{ren}}(q^2)$, the most important quantity in the calculation of virtual gluon corrections to the $\tau(P) \to \nu_\tau(p_1) + q_2(p_2) + \overline{q}_3(p_3)$ decay rate. As noted before, to order g_{s}^2 correction to the rate, we need the sum – denoted by $\mathcal{M}_{\mathrm{Vi}}$ – of the tree amplitude of Fig. 14.1 with the renormalized loop amplitude of Fig. 14.2. Their interference in the square $|\mathcal{M}_{\mathrm{Vi}}|^2$ is $\mathcal{O}(g_{\mathrm{s}}^2)$:

$$\mathcal{M}_{\mathrm{Vi}} = \frac{G_{\mathrm{F}} V_{\mathrm{q_2 q_3}}}{\sqrt{2}} \overline{u}(p_1)\gamma_\mu(1-\gamma_5)u(P)\Big\{\overline{u}(p_2)[\gamma^\mu(1-\gamma_5) + \widetilde{\Gamma}^\mu_{\mathrm{ren}}(p_2,p_3)]v(p_3)\Big\}\ .$$

To lighten the computations of $\Gamma^\mu(p_2,p_3)$, hence of $\widetilde{\Gamma}^\mu_{\mathrm{ren}}(p_2,p_3)$ or $\mathcal{M}_{\mathrm{Vi}}$, we assume from now on $m=0$. With $m=0$, the expression for $\Gamma^\mu(p_2,p_3)$ is considerably simpler. The vector current is conserved, and of course the Ward identity holds in (42). Going back to (12), we see that only $F_1(q^2) = G_1(q^2)$ survives, while $F_2(q^2) = G_3(q^2) = 0$. Then

$$\widetilde{\Gamma}^\mu_{\mathrm{ren}}(p_2,p_3) = \gamma^\mu(1-\gamma_5)\widetilde{F}_1^{\mathrm{ren}}(q^2)\ .$$

Until now, we have postponed the problem of infrared divergence (IR) which occurs in $F_1(q^2)$, hence in $\widetilde{F}_1^{\mathrm{ren}}(q^2)$. Now is the time to treat it together with the real gluon emission process, also called bremsstrahlung.

The second part of $F_1(q^2)$ in the second line of (12) – that without the singular $\Gamma(2-\frac{n}{2})$ – is free of UV divergence. It has only IR divergence. To handle the latter, since with $\int_0^1 \mathrm{d}u/u$ in (19) we can go no further, we may try to parameterize the IR of $F_1(q^2)$ again as poles in ε, on equal footing with UV.

This can be done by going back to our original (9) and keeping everywhere the $\Gamma(3-\frac{n}{2})$ which is equal to 1 for $n=4$.

The denominator $-\mathcal{D}(x,y) = m^2(x+y)^2 - q^2xy$ on the second line of (12) is always written under the original form $[-\mathcal{D}(x,y)]^{3-\frac{n}{2}}$ as in (9), i.e. we keep $\varepsilon \neq 0$ even if the UV divergences are absent because we would like to parameterize the IR divergences by the ε pole too. As for the first part of $F_1(q^2)$ appearing on the first line of (12) with $\Gamma(2-\frac{n}{2})$, we rewrite it in terms of $\Gamma(3-\frac{n}{2})$ using $\Gamma(2-\frac{n}{2}) = 2\,\Gamma(3-\frac{n}{2})/(4-n)$. With the relations

$$\int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \ , \ \text{ and } \ z\,\Gamma(z) = \Gamma(1+z) \ , \tag{14.49}$$

we get altogether from (12), with $m=0$,

$$\begin{aligned} F_1(q^2) &= \frac{4}{3}\frac{g_\mathrm{s}^2}{2^n\pi^{n/2}}\frac{\Gamma(3-\frac{n}{2})[\Gamma(\frac{n}{2}-1)]^2}{\Gamma(n-2)}\left[\frac{-8}{(4-n)^2}+\frac{2}{4-n}-2\right]\left(-q^2\right)^{\frac{n}{2}-2} \ , \\ &\equiv g_\mathrm{s}^2\,\mathcal{A}(n)\ \left(+q^2\right)^{\frac{n}{2}-2}\ \exp\left(\mathrm{i}\,\frac{n\,\pi}{2}\right) \ , \end{aligned} \tag{14.50}$$

where

$$\mathcal{A}(n) = \frac{4}{3}\,\frac{\Gamma(3-\frac{n}{2})[\Gamma(\frac{n}{2}-1)]^2}{\Gamma(n-2)\,2^n\pi^{n/2}}\left[\frac{-8}{(4-n)^2}+\frac{2}{4-n}-2\right] \ .$$

Due to the masslessness of both the gluon and the fermion, the double pole $1/\varepsilon^2$ in (50) has a pure IR origin. It comes from

$$\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1}{(x\,y)^{3-\frac{n}{2}}} = \frac{4}{(4-n)^2}\,\frac{[\Gamma(\frac{n}{2}-1)]^2}{\Gamma(n-3)} \ .$$

Since we are free to continue analytically any result to $n\geq 4$, from (50) we find that $F_1(0)=0$, and (38) becomes $\widetilde{\Gamma}^\mu_\mathrm{ren}(p_2,p_3) = \gamma^\mu(1-\gamma_5)F_1(q^2)$. Then

$$\begin{aligned} \mathcal{M}_\mathrm{Vi} = &\,\frac{G_\mathrm{F}V_{\mathrm{q_2q_3}}}{\sqrt{2}}\overline{u}(p_1)\gamma_\mu(1-\gamma_5)u(P) \\ &\times\left\{\overline{u}(p_2)\gamma^\mu(1-\gamma_5)\left[1+F_1(q^2)\right]\,v(p_3)\right\} \ . \end{aligned} \tag{14.51}$$

Remark. As $F_1(0) = 0 = \delta_\mathrm{q}$, one may suspect that the UV divergence in $F_1(q^2)$ could not be removed since $\widetilde{F}_1^\mathrm{ren}(q^2) = F_1(q^2)$. In fact, $F_1(0)$ vanishes only in the approximation $m=0$. However, as explicitly written in (20),

$F_1(0)$ contains both UV and IR divergences, its vanishing is equivalent to the cancelation of a UV by an IR. Since these two kinds of divergence are parameterized by the same $4-n$ poles, the simple pole $2/(4-n)$ in (50) is understood as coming from an IR divergence which replaces a UV, the latter is implicitly removed by the interchange of the UV and IR poles which is imposed by $F_1(0)=0$. In other words, all the poles in (50) represent IR. ■

For timelike $q^2 > 4m^2$, the form factor $F_1(q^2)$ develops its imaginary part, represented by $\exp(\mathrm{i}\,\frac{n\pi}{2})$. Putting the expression of $F_1(q^2)$ in (50) into $\mathcal{M}_{\mathrm{Vi}}$ in (51), after averaging the spin of the initial τ lepton state, as well as summing the spins and colors of the final states, we obtain *to order* g_{s}^2

$$\frac{1}{2}\sum_{\text{spin,color}} |\mathcal{M}_{\mathrm{Vi}}|^2 = g_{\mathrm{s}}^2\,\mathcal{B}(q^2)\,\mathrm{Tr}[\not{p}_1\gamma_\mu\,\not{P}\gamma_\nu(1-\gamma_5)]\;\mathrm{Tr}[\not{p}_2\gamma^\mu\,\not{p}_3\gamma^\nu(1-\gamma_5)]\,,$$
$$\mathcal{B}(q^2) \equiv N_c\,G_{\mathrm{F}}^2|V_{\mathrm{q_2q_3}}|^2\left\{2\mathcal{A}(n)\,\cos\frac{n\pi}{2}\right\}\left(q^2\right)^{\frac{n}{2}-2}\,. \tag{14.52}$$

This g_{s}^2 order of $|\mathcal{M}_{\mathrm{Vi}}|^2$ is the interference between the tree amplitude of Fig. 14.1 and the renormalized loop amplitude of Fig. 14.2. Having obtained $|\mathcal{M}_{\mathrm{Vi}}|^2$ in (52), we go on to compute the decay rate Γ_{Vi} by using the formulas of Chap. 4 as in 4 dimensions, with only the replacement of 4 by n for the phase space integral. Thus,

$$\Gamma_{\mathrm{Vi}} = \frac{1}{2M}\,g_{\mathrm{s}}^2\int\frac{\mathrm{d}^{n-1}p_1}{2E_1\,(2\pi)^{n-1}}\,\mathcal{B}(q^2)\,\mathrm{Tr}[\not{p}_1\gamma_\mu\,\not{P}\gamma_\nu(1-\gamma_5)]\,\frac{(2\pi)^n}{(2\pi)^{2n-2}}$$
$$\times\int\!\!\int\frac{\mathrm{d}^{n-1}p_2}{2E_2}\frac{\mathrm{d}^{n-1}p_3}{2E_3}\,\mathrm{Tr}[\not{p}_2\gamma^\mu\,\not{p}_3\gamma^\nu(1-\gamma_5)]\delta^{(n)}(p_2+p_3-q)\,. \tag{14.53}$$

The double integration $\int\!\int \mathrm{d}^{n-1}p_2\;\mathrm{d}^{n-1}p_3$ of (53) is symmetric by interchanging p_2 and p_3. Therefore the contribution of the γ_5 in the integrand $\mathrm{Tr}[\not{p}_2\gamma^\mu\,\not{p}_3\gamma^\nu(1-\gamma_5)]$ vanishes because of its p_2–p_3 antisymmetric character. Only terms symmetric in the μ–ν permutation remain. In turn, it renders the γ_5 of $\mathrm{Tr}[\not{p}_1\gamma_\mu\,\not{P}\gamma_\nu(1-\gamma_5)]$ on the first line of (53) superfluous.

We compute now the double integration of (53) denoted by $J^{\mu\nu}(q^2)$:

$$J^{\mu\nu}(q^2) \equiv \int\!\!\int\frac{\mathrm{d}^{n-1}p_2}{2E_2}\,\frac{\mathrm{d}^{n-1}p_3}{2E_3}\,\mathrm{Tr}[\not{p}_2\gamma^\mu\,\not{p}_3\gamma^\nu]\,\delta^{(n)}(p_2+p_3-q)\,. \tag{14.54}$$

First we notice that $q_\mu J^{\mu\nu}(q^2)=0$ because of $m^2=p_2^2=p_3^2=0$, implying that $J^{\mu\nu}(q^2)$ must have the structure $J^{\mu\nu}(q^2)=(-g^{\mu\nu}q^2+q^\mu q^\nu)L(q^2)$.

Since $J^{\mu\nu}(q^2)$ is a function of q^2, it is convenient to use the center-of-mass frame where $\boldsymbol{q}=\boldsymbol{p}_2+\boldsymbol{p}_3=\mathbf{0}$. The variable q^μ is $(\sqrt{q^2},\boldsymbol{q}=\mathbf{0})$ and $2E_2=2E_3=\sqrt{q^2}$. We multiply the left and the right sides of (54) by $g_{\mu\nu}$, and integrate over $\mathrm{d}^{n-1}p_3$ to get rid of $\delta^{n-1}(p_2+p_3-q)$. The result is

$$(1-n)L(q^2) = 2(2-n)\int\frac{\mathrm{d}^{n-1}p_2}{2E_2}\,\frac{1}{2E_3}\,\delta(E_2+E_3-\sqrt{q^2})\,. \tag{14.55}$$

As remarked before, $\mathrm{Tr}[\not{p}_2\gamma^\mu \not{p}_3\gamma^\nu] = 4\,[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2\cdot p_3)g^{\mu\nu}]$ for $n \neq 4$ in (54). The n-dimensional solid angle is defined by $n-1$ angles: $\theta,\ \theta_1, \cdots, \theta_{n-2}$,

$$\Omega^n = \int \mathrm{d}\Omega^n = \int_0^\pi \mathrm{d}\theta(\sin\theta)^{n-2} \int_0^\pi \mathrm{d}\theta_1(\sin\theta_1)^{n-3} \cdots \int_0^{2\pi} \mathrm{d}\theta_{n-2}\ .$$

Using (49), we now demonstrate the following useful formulas

$$\begin{aligned}
I_k &\equiv \int_0^\pi \mathrm{d}\theta(\sin\theta)^k = \int_{-1}^{+1} \mathrm{d}\cos\theta(1-\cos^2\theta)^{\frac{k-1}{2}} = \int_0^1 \mathrm{d}x\ x^{-\frac{1}{2}}\,(1-x)^{\frac{k-1}{2}} \\
&= \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(k+1)]}{\Gamma[\frac{1}{2}(k+2)]}\ , \quad I_0 = \pi \ \longrightarrow\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\ ,
\end{aligned}$$

$$\Omega^n = \left\{\frac{\sqrt{\pi}\,\Gamma[\frac{1}{2}(n-1)]}{\Gamma(\frac{n}{2})}\right\} \cdots \left\{\frac{\sqrt{\pi}\Gamma(\frac{1}{2}[n-(n-2)])}{\Gamma(\frac{1}{2}[n-(n-3)])}\right\} \times (2\pi) = \frac{2\,\pi^{n/2}}{\Gamma(\frac{n}{2})}. \quad (14.56)$$

With (56), the quantity $\mathrm{d}^{n-1}p_2/2E_2$ in (55) can be written as

$$\frac{\mathrm{d}^{n-1}p_2}{2E_2} = \frac{E_2^{n-3}\,\mathrm{d}E_2}{2}\ \Omega^{n-1} = E_2^{n-3}\ \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\ \mathrm{d}E_2\ . \quad (14.57)$$

Putting (57) back into (55), we integrate over E_2 to eliminate the last function $\delta(2E_2 - \sqrt{q^2})$. Using $(n-1)\Gamma[\frac{1}{2}(n-1)] = 2\,\Gamma[\frac{1}{2}(n+1)]$, we obtain

$$J^{\mu\nu}(q^2) = \frac{(n-2)\ \pi^{\frac{n-1}{2}}}{2^{n-2}\ \Gamma(\frac{n+1}{2})}\ (q^2)^{\frac{n}{2}-2}\ \left[-q^2\,g^{\mu\nu} + q^\mu q^\nu\right]\ . \quad (14.58)$$

We now insert the above expression of $J^{\mu\nu}(q^2)$ into (53); the product of $\mathrm{Tr}[\not{p}_1\gamma_\mu \not{P}\gamma_\nu]$ with the tensor $[-q^2\,g^{\mu\nu} + q^\mu q^\nu]$ is found to be

$$[-q^2\,g^{\mu\nu} + q^\mu q^\nu]\ [\mathrm{Tr}[\not{p}_1\gamma_\mu \not{P}\gamma_\nu] = 2M^4(1-\xi)[1+(n-2)\xi]\,, \quad (14.59)$$

where $\xi = q^2/M^2$. The remaining integral $\mathrm{d}^{n-1}p_1$ in (53) is simple in the rest frame of the decaying τ lepton. Using $(P-p_1)^2 = (M^2 - 2ME_1) = q^2$,

$$\begin{aligned}
\int \frac{\mathrm{d}^{n-1}p_1}{2E_1\,(2\pi)^{n-1}} &= \Omega^{n-1} \int \frac{E_1^{n-2}\,\mathrm{d}E_1}{2E_1\,(2\pi)^{n-1}} = \int \frac{E_1^{n-3}\,\mathrm{d}E_1}{\Gamma(\frac{n-1}{2})\ (4\pi)^{\frac{n-1}{2}}} \\
&= \frac{M^{n-2}}{2^{2n-3}\,\pi^{\frac{n-1}{2}}\ \Gamma(\frac{n-1}{2})} \int_0^1 \mathrm{d}\xi\ (1-\xi)^{n-3}\ . \quad (14.60)
\end{aligned}$$

Putting together (52)–(60), we get

$$\begin{aligned}
\frac{\Gamma_{\mathrm{Vi}}}{\Gamma_0} =& \frac{\left[N_c|V_{\mathrm{q_2q_3}}|^2\ \alpha_{\mathrm{s}}\right]\ 2^{5\varepsilon}\ \pi^{\frac{3}{2}\varepsilon}\ \Gamma(2-\frac{1}{2}\varepsilon)}{M^{3\varepsilon}\Gamma[\frac{1}{2}(3-\varepsilon)]\,\Gamma[\frac{1}{2}(5-\varepsilon)]} \int_0^1 \mathrm{d}\xi\,\xi^{-\varepsilon}\,(1-\xi)^{2-\varepsilon}[1+(2-\varepsilon)\,\xi] \\
&\times \left\{\frac{\Gamma(1+\frac{1}{2}\varepsilon)\Gamma(1-\frac{1}{2}\varepsilon)}{\Gamma(2-\varepsilon)} \cos(\tfrac{1}{2}\pi\varepsilon) \left[-\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon} - 1\right]\right\}\ , \quad (14.61)
\end{aligned}$$

where $\Gamma_0 = G_{\rm F}^2 M^5/192\pi^3$ already given in (13.21) is the QCD uncorrected decay rate $\tau^- \to \nu_\tau + {\rm q}_2 + \overline{\rm q}_3$ of Fig. 14.1, in which colors (of massless quarks) are not yet summed, i.e. the factor $N_{\rm c}$ is not yet included.

As we will see, the IR divergent *singular* ε pole terms in (61) are exactly canceled by those of the bremsstrahlung rate that we are going now to compute from the two diagrams in Fig. 14.3, such that the sum of (61) and (81) is *infrared convergent*. We remark further that both (61) and (81) share a common regular term represented by the first factor, i.e. the first line on the right hand side of (61). Therefore, we keep untouched this factor and we only expand up to $\mathcal{O}(\varepsilon^2)$ the regular terms $\Gamma(1+\frac{1}{2}\varepsilon)\Gamma(1-\frac{1}{2}\varepsilon)/\Gamma(2-\varepsilon)]\times\cos(\frac{1}{2}\pi\varepsilon)$ in the second factor (second line) represented by the curly brackets $\{\}$ of (61). When these ε^2 expansions are multiplied by the poles $-4/\varepsilon^2 + 1/\varepsilon - 1$ in $\{\}$, some finite terms emerge. This expansion of the regular terms up to second order in ε is therefore mandatory. We obtain for the curly brackets [second line of (61)] the following result

$$\{\} = -\frac{4}{\varepsilon^2} + \frac{4\gamma_E - 3}{\varepsilon} - 2\gamma_E^2 + 3\gamma_E - 4 + \frac{2\pi^2}{3} + \mathcal{O}(\varepsilon)\ , \tag{14.62}$$

using (49) together with the expansion up to z^2 of $\Gamma(1+z)$ for $z \ll 1$,

$$\Gamma(1+z) \underset{z\to 0}{\longrightarrow} 1 - \gamma_E\, z + \frac{6\gamma_E^2 + \pi^2}{12}\, z^2 + \mathcal{O}(z^3)\ . \tag{14.63}$$

14.5 Real Gluon Contributions

The amplitude of the two diagrams in Fig. 14.3 can be written as

$$\begin{aligned}\mathcal{M}_{\rm Re} =& \frac{G_{\rm F}\, V_{{\rm q}_2{\rm q}_3}\ g_{\rm s}}{\sqrt{2}}\ \overline{u}(p_1)\gamma_\mu\,(1-\gamma_5)u(P) \\ &\times \overline{u}(p_2)\left[\not{\varepsilon}_{k'}\ \frac{\lambda_i}{2}\ \frac{\not{p}_2 + \not{k}'}{(p_2+k')^2}\gamma^\mu - \gamma^\mu\ \frac{\not{p}_3 + \not{k}'}{(p_3+k')^2}\ \not{\varepsilon}_{k'}\ \frac{\lambda_i}{2}\right](1-\gamma_5)\ v(p_3).\end{aligned} \tag{14.64}$$

In (64), $\varepsilon^\alpha(k', i)$ is the gluon polarization vector associated with the eight SU(3) color matrices λ_i. To simplify the trace calculation of many γ matrices, we write $\not{\varepsilon}_{k'}\ \not{p}_2 = -\not{p}_2\ \not{\varepsilon}_{k'} + 2p_2\cdot\varepsilon_{k'}$, and $\not{p}_3\ \not{\varepsilon}_{k'} = -\not{\varepsilon}_{k'}\ \not{p}_3 + 2p_3\cdot\varepsilon_{k'}$, and apply the Dirac equation to the spinors $\overline{u}(p_2)$, $v(p_3)$ in (64). The transition probability $|\mathcal{M}_{\rm Re}|^2$, summed and averaged over spins and colors in the usual manner, can best be expressed in terms of a tensor $\mathcal{T}^{\mu\nu}(p_2, p_3, k')$ defined below. We find

$$\begin{aligned}&\tfrac{1}{2}\sum_{\rm color,spins}|\mathcal{M}_{\rm Re}|^2 = g_{\rm s}^2\ \frac{N_{\rm c}}{3}G_{\rm F}^2\ |V_{{\rm q}_2{\rm q}_3}|^2\ {\rm Tr}[\not{p}_1\gamma_\mu\ \not{P}\gamma_\nu(1-\gamma_5)]\ \mathcal{T}^{\mu\nu}(p_2,p_3,k'), \\ &\mathcal{T}^{\mu\nu}(p_2,p_3,k') = {\rm Tr}\Bigg\{\not{p}_2\left[\frac{2p_2^\alpha + \gamma^\alpha\ \not{k}'}{p_2\cdot k'}\gamma^\mu - \gamma^\mu\frac{2p_3^\alpha + \not{k}'\gamma^\alpha}{p_3\cdot k'}\right] \\ &\times \not{p}_3\left[\gamma^\nu\frac{2p_2^\beta + \not{k}'\gamma^\beta}{p_2\cdot k'} - \frac{2p_3^\beta + \gamma^\beta\ \not{k}'}{p_3\cdot k'}\gamma^\nu\right](1-\gamma_5)\Bigg\}(-g_{\alpha\beta}),\end{aligned} \tag{14.65}$$

the factor $-g_{\alpha\beta}$ comes from the summation over the gluon polarizations. Using the fact that the bremsstrahlung rate is obtained by integration over the symmetric phase space in p_2, p_3, k', it is easy to show that the γ_5 can be dropped from $\mathcal{T}^{\mu\nu}(p_2, p_3, k')$. Then the integration of the latter will result in a term symmetric in μ and ν. This in turn renders irrelevant the γ_5 in $\mathrm{Tr}[\not{p}_1\gamma_\mu \not{P}\gamma_\nu(1-\gamma_5)]$ of (65). Thus both γ_5 can be eliminated.

14.5.1 Infrared Divergence

As explicitly shown in $\mathcal{T}^{\mu\nu}(p_2, p_3, k')$, the denominators $p_2 \cdot k'$ and $p_3 \cdot k'$ indicate that the real gluon emission rate is divergent in the limit where the energy-momentum k' of the gluon tends to zero. For massless quarks, these denominators vanish also when both the gluon and the quark are emitted in parallel, regardless of the gluon energy. In these limits, the processes with radiated gluons cannot be distinguished from those without gluons. The bremsstrahlung is thus an essential part of radiative corrections in this τ decay as well as in all other QCD (QED) reactions with real gluons (photons) emitted. The tensor $\mathcal{T}^{\mu\nu}(p_2, p_3, k')$ is found to be

$$\begin{aligned}\frac{1}{4}\mathcal{T}^{\mu\nu}(p_2,p_3,k') = &-g^{\mu\nu}\left[\frac{8s^2}{ut}+\frac{8s}{u}+\frac{8s}{t}+2(n-2)\left(\frac{u}{t}+\frac{t}{u}\right)+4(n-4)\right]\\ &- p_2^\mu p_2^\nu\left[\frac{16}{t}\right]-p_3^\mu p_3^\nu\left[\frac{16}{u}\right]-k'^\mu k'^\nu\left[\frac{8(n-4)s}{ut}\right]\\ &+(p_2^\mu p_3^\nu+p_2^\nu p_3^\mu)\left[\frac{16s}{ut}+\frac{8}{u}+\frac{8}{t}\right]\\ &+(p_2^\mu k'^\nu+p_2^\nu k'^\mu)\left[\frac{8s}{ut}+\frac{4(n-4)}{t}+\frac{4(n-2)}{u}\right]\\ &+(p_3^\mu k'^\nu+p_3^\nu k'^\mu)\left[\frac{8s}{ut}+\frac{4(n-4)}{u}+\frac{4(n-2)}{t}\right],\end{aligned} \quad (14.66)$$

where the three invariants s, t, u are defined as follows, only two of them are independent on account of the momentum conservation

$$s = 2\,p_2\cdot p_3 \ \ , \ \ t = 2\,p_2\cdot k' \ \ , \ \ u = 2\,p_3\cdot k' \ \ , \ \ s+t+u = q^2 \ .$$

For $m_2 = m_3 = m$, and *a fortiori* for $m = 0$, the conservation of the vector current at the vertex $W\mathrm{q}_2\overline{\mathrm{q}}_3$ in Fig. 14.3 implies that $q_\mu\mathcal{T}^{\mu\nu}(p_2, p_3, k') = q_\nu\mathcal{T}^{\mu\nu}(p_2, p_3, k') = 0$, where $q = p_2 + p_3 + k'$. We can verify this property by multiplying the right-hand side of (66) by $(p_2 + p_3 + k')_\mu$, and check that the product effectively is equal to zero.

Using (65), the decay width $\Gamma_{\mathrm{Re}} \equiv \Gamma(\tau \to \nu_\tau + \mathrm{q}_2 + \overline{\mathrm{q}}_3 + \mathrm{g})$ is

$$\begin{aligned}\Gamma_{\mathrm{Re}} =&\frac{1}{2M}\frac{N_{\mathrm{c}}}{3}G_{\mathrm{F}}^2\,|V_{\mathrm{q_2q_3}}|^2\,g_{\mathrm{s}}^2\int\frac{\mathrm{d}^{n-1}p_1}{(2\pi)^{n-1}2E_1}\mathrm{Tr}[\not{p}_1\gamma_\mu \not{P}\gamma_\nu]\\ &\times\frac{(2\pi)^n}{(2\pi)^{3n-3}}\int_{\mathcal{PS}3}\mathcal{T}^{\mu\nu}(p_2,p_3,k')\ ,\end{aligned} \quad (14.67)$$

where the three-body phase space integral is denoted by $\int_{\mathcal{PS}3}$,

$$\int_{\mathcal{PS}3} \equiv \int \frac{\mathrm{d}^{n-1}p_2}{2E_2} \frac{\mathrm{d}^{n-1}p_3}{2E_3} \frac{\mathrm{d}^{n-1}k'}{2E_{k'}} \delta^n(p_2 + p_3 + k' - q) \,. \tag{14.68}$$

No matter how complicated the integration $\int_{\mathcal{PS}3} \mathcal{T}^{\mu\nu}(p_2, p_3, k')$ is, it results only in a function of the momentum transfer q^2. Moreover, this phase space integration $\int_{\mathcal{PS}3} \mathcal{T}^{\mu\nu}(p_2, p_3, k')$ must have the structure $-q^2\, g^{\mu\nu} + q^\mu q^\nu$, due to $q_\mu \mathcal{T}^{\mu\nu}(p_2, p_3, k') = q_\nu \mathcal{T}^{\mu\nu}(p_2, p_3, k') = 0$, so that

$$\int_{\mathcal{PS}3} \mathcal{T}^{\mu\nu}(p_2, p_3, k') = (-q^2\, g^{\mu\nu} + q^\mu q^\nu) H(q^2) \,. \tag{14.69}$$

To compute $H(q^2)$, we multiply the left- and the right-hand sides of (69) by $g_{\mu\nu}$. Using the expression of $\mathcal{T}^{\mu\nu}(p_2, p_3, k')$ in (66), we get

$$q^2 H(q^2) = \frac{8(2-n)}{1-n} \int_{\mathcal{PS}3} \left\{ 4 \left[\frac{s^2}{ut} + \frac{s}{u} + \frac{s}{t} \right] + (n-2) \left[\frac{u}{t} + \frac{t}{u} \right] + 2(n-4) \right\} .$$

Since $\int_{\mathcal{PS}3}$ is completely symmetric in the three integration variables $p_2,\ p_3$, and k' and all of these three particles are massless, the integration of the three variables s, t, and u are completely equivalent because of the possible interchange among $p_2,\ p_3$, and k'. More precisely, we have

$$\int_{\mathcal{PS}3} \left(\frac{u}{t} \right) = \int_{\mathcal{PS}3} \left(\frac{t}{u} \right) \,.$$

Then using $(s + t + u) = q^2$, we obtain a simple form for $H(q^2)$:

$$H(q^2) = \frac{16}{q^2} \frac{n-2}{n-1} \int_{\mathcal{PS}3} \left[\frac{2\, sq^2}{ut} + (n-2)\frac{u}{t} + (n-4) \right] \,. \tag{14.70}$$

Our next task is to evaluate the three-body phase space integral $\int_{\mathcal{PS}3}$.

14.5.2 Three-Particle Phase Space

It is instructive to see how $\int_{\mathcal{PS}3}$ can be decomposed and computed, first in $n = 4$ dimensions. By energy-momentum conservation, there are in all $3 \times 3 - 4 = 5$ independent variables that describe the three-body phase space. Since $\int_{\mathcal{PS}3}$ is a function of q^2, it may be convenient to choose the rest frame of q^μ , i.e. $\boldsymbol{q} = \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{k}' = \boldsymbol{0}$: $q^\mu = (\sqrt{q^2}, \boldsymbol{0})$. The three momenta $\boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{k}'$ define a plane $\mathcal{P}$ in this frame. The vector $\boldsymbol{k}'$ is fixed by E_2, E_3 and the angle θ between $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$. Energy conservation restricts θ in terms of E_2, E_3. So only E_2 and E_3 are independent. Thus for the three massless particles:

$$\begin{aligned} E_{k'} &= \sqrt{E_2^2 + E_3^2 + 2E_2\, E_3 \cos\theta} \ , \\ 2E_2\, E_3 \cos\theta &= q^2 + 2E_2\, E_3 - 2\sqrt{q^2}(E_2 + E_3) \,. \end{aligned} \tag{14.71}$$

The three remaining independent variables can be chosen as the angular orientation of $\boldsymbol{p}_2, \boldsymbol{p}_3$, and $\boldsymbol{k}'$. For instance, two angles, denoted by Ω to determine the vector $\boldsymbol{p}_2$, and one angle Φ to fix the plane $\mathcal{P}$ around $\boldsymbol{p}_2$. Using $\int \mathrm{d}^3k'\ \delta^3(p_2+p_3+k'-q)=1$, we have for $m_2=m_3=0$,

$$\int_{\mathcal{PS}3}(n=4) = \int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\Phi \int \frac{E_2 \mathrm{d}E_2}{2} \int \frac{E_3 \mathrm{d}E_3}{2} \times \int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{2E_{k'}} \delta\left(E_2+E_3+E_{k'}-\sqrt{q^2}\right) . \tag{14.72}$$

We remove the last δ-function by performing the $\cos\theta$ integration, using

$$\delta\{f(x)\} = \frac{\delta(x-x_0)}{|f'(x)|_{x=x_0}} \ \to\ \delta\left(E_2+E_3+E_{k'}-\sqrt{q^2}\right) = \frac{\delta(\cos\theta - z)}{\mathrm{d}E_{k'}/\mathrm{d}\cos\theta|_z} ,$$

where $z=\cos\theta_0$ is solution to (71) for fixed E_2, E_3. From (71), we have

$$\frac{\mathrm{d}E_{k'}}{\mathrm{d}\cos\theta} = \frac{E_2E_3}{E_{k'}} \Longrightarrow \int \frac{\mathrm{d}\cos\theta}{2E_{k'}} \delta\left(E_2+E_3+E_{k'}-\sqrt{q^2}\right) = \frac{1}{2E_2E_3} , \tag{14.73}$$

and so we obtain

$$\int_{\mathcal{PS}3}(n=4) = \frac{1}{2^3}\int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\Phi \int_\triangle \mathrm{d}E_2\ \mathrm{d}E_3 = \pi^2 \int_\triangle \mathrm{d}E_2\ \mathrm{d}E_3 . \tag{14.74}$$

Due to (71), the E_2 and E_3 integration domain $\triangle$ is restricted by

$$|q^2 + 2E_2\,E_3 - 2\sqrt{q^2}(E_2+E_3)| \le 2E_2E_3 , \tag{14.75}$$

which is translated into a rectangular isosceles triangle limited by three lines in the (E_2, E_3) plane: $E_2 = \frac{1}{2}\sqrt{q^2}$, $E_3 = \frac{1}{2}\sqrt{q^2}$, and $E_2+E_3 = \frac{1}{2}\sqrt{q^2}$.

A digression. In the general case of a particle A decaying into 3 others, say $a_1+a_2+a_3$, the squared decay amplitude multiplied by the phase space volume (72) will give the double distribution $\mathrm{d}\Gamma/(\mathrm{d}E_j\ \mathrm{d}E_\ell)$ of a_j, a_ℓ energies. The kinematic result in (74) indicates that a plot of $\mathrm{d}\Gamma/(\mathrm{d}E_j\ \mathrm{d}E_\ell)$ is a powerful tool for investigating the decay dynamics, as suggested by Dalitz.

Indeed, if the amplitude is constant, the events will be uniformly distributed in the $\triangle$ domain according to (74). On the other hand, any dynamical specific structure of the amplitude will be immediately revealed by a characteristic density of events in this plot. A concentration of events which cluster along a line in the $\triangle$ domain of E_j, E_ℓ corresponds to the presence of a resonance formed by a_j and a_ℓ: $A \to a_k + B$ follows by $B \to a_j + a_\ell$. Many hadronic resonances are found by this method. The angular Ω and Φ distributions $\mathrm{d}\Gamma/(\mathrm{d}\Omega\mathrm{d}\Phi)$, on the other hand, provide useful information on the spin and intrinsic parity of the decaying particle A, once the decay amplitude squared is incorporated in (72). ■

14.5.3 Bremsstrahlung Rate

Going back to our case of (68) and (70), we first integrate over the gluon momentum $\mathrm{d}^{n-1}k'$ to get rid of the $\delta^{n-1}(p_2+p_3+k'-q)$, similar to (72). Together with (56) and (57), we write

$$\frac{\mathrm{d}^{n-1}p_3}{2E_3} = \frac{E_3^{n-3}\mathrm{d}E_3}{2}\,\Omega^{n-2}\,\mathrm{d}\theta(\sin\theta)^{n-3}\ ,\ \text{with}\ \ \Omega^{n-2} = \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})}\ ,$$
$$\mathrm{d}\theta(\sin\theta)^{n-3} = \mathrm{d}\cos\theta\,(1-\cos\theta)^{\frac{n}{2}-2}\,(1+\cos\theta)^{\frac{n}{2}-2}\ . \tag{14.76}$$

From the isosceles triangle in (75), one has $\sqrt{q^2} \le 2(E_2+E_3) \le 2\sqrt{q^2}$, it proves convenient to introduce two variables x, y related to E_2, E_3 by

$$E_2 = \frac{\sqrt{q^2}\,x}{2}\quad,\quad E_3 = \frac{\sqrt{q^2}(1-xy)}{2}\quad,\quad \mathrm{d}E_2\,\mathrm{d}E_3 = \frac{q^2}{4}x\,\mathrm{d}x\,\mathrm{d}y\ , \tag{14.77}$$

then from (71), we get

$$1-\cos\theta = \frac{2(1-y)}{1-xy}\quad,\quad 1+\cos\theta = \frac{2y(1-x)}{1-xy}\ . \tag{14.78}$$

Using (73), the remaining $\delta(E_2+E_3+E_{k'}-\sqrt{q^2})$ is replaced by $E_{k'}/(E_2E_3)$ after the $\cos\theta$ integration. With (57), (76), (77), and (78), the n-dimension three-body phase space $\int_{\mathcal{PS}3}$ defined in (68) can now be written as

$$\int_{\mathcal{PS}3} = \frac{\pi^{n-\frac{3}{2}}}{2^{n-1}}\frac{[q^2]^{n-3}}{\Gamma[\frac{1}{2}(n-1)]\,\Gamma[(\frac{1}{2}(n-2)]}$$
$$\times\int_0^1 \mathrm{d}x\,x^{n-3}\,(1-x)^{\frac{n}{2}-2}\int_0^1 \mathrm{d}y\,[y\,(1-y)]^{\frac{n}{2}-2}\ . \tag{14.79}$$

Using (77) and (78), we have

$$s = q^2\,x\,(1-y)\quad,\quad t = q^2x\,y\quad,\quad u = q^2\,(1-x)\ . \tag{14.80}$$

With (79) and (80) plugged into (70) and using (49), we finally get

$$H(q^2) = \frac{[q^2]^{n-4}\pi^{n-\frac{3}{2}}}{2^{n-5}}\,\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{3n}{2}-3)}\left\{\frac{2n(n-2)}{(n-4)^2}+\frac{n^2-4}{(n-4)}+(n-4)\right\}\ .$$

The $\varepsilon = 4-n$ double and simple poles in the above equation come from the integration of $2sq^2/ut$ and $(n-2)u/t$ of (70) respectively.

Putting the quantity $H(q^2)$ back into (69) and (67) and using again (59) and (60), we obtain the bremsstrahlung rate

$$\frac{\Gamma_{\mathrm{Re}}}{\Gamma_0} = \frac{\left[\mathrm{N_c}|V_{\mathrm{q_2q_3}}|^2\,\alpha_{\mathrm{s}}\right]\ 2^{5\varepsilon}\,\pi^{\frac{3}{2}\varepsilon}\,\Gamma(2-\frac{1}{2}\varepsilon)}{M^{3\varepsilon}\Gamma[\frac{1}{2}(3-\varepsilon)]\,\Gamma[\frac{1}{2}(5-\varepsilon)]}\int_0^1 \mathrm{d}\xi\,\xi^{-\varepsilon}\,(1-\xi)^{2-\varepsilon}[1+(2-\varepsilon)\,\xi]$$
$$\times\frac{\Gamma(1-\frac{1}{2}\varepsilon)}{\Gamma(3-\frac{3}{2}\varepsilon)}\left[\frac{8}{\varepsilon^2}-\frac{12}{\varepsilon}+5\right]\ . \tag{14.81}$$

The first factor on the first line of (81) is exactly the same as the first factor of (61) for the virtual gluon correction rate, so we only need to expand the regular terms in the second line of (81) up to ε^2. Using (63), we find

$$\frac{\Gamma(1-\frac{1}{2}\varepsilon)}{\Gamma(3-\frac{3}{2}\varepsilon)}\left[\frac{8}{\varepsilon^2}-\frac{12}{\varepsilon}+5\right]=\frac{4}{\varepsilon^2}-\frac{4\gamma_E-3}{\varepsilon}+2\gamma_E^2-3\gamma_E+\frac{19}{4}-\frac{2\pi^2}{3}\,. \quad (14.82)$$

14.6 Final Result

As explicitly shown, the ε poles of (82) exactly cancel those of (62), i.e. the sum of the right-hand sides of (62) and (82) is finite and equal to $-4+\frac{19}{4}$. The sum of virtual and real gluon corrections to the rate is now free of IR divergences, so we put $\varepsilon = 0$ in the first line of Γ_{Vi} and Γ_{Re} in (61) and (81).

Let us summarize. The ultraviolet divergences of loop diagrams in Fig. 14.2 are removed by replacing $F_1(q^2)$ with the renormalized form factor $F_1^{\mathrm{ren}}(q^2) = F_1(q^2) - F_1(0)$. Both $F_1(q^2)$ and $F_1(0)$ are UV divergent, but their difference is free of UV divergences. The next step deals with the infrared divergences in both Γ_{Vi} and Γ_{Re}. They are found to cancel exactly each other to yield a finite result free of both UV and IR divergences. The final result for the radiative corrections is the sum of (61) and (81)

$$\begin{aligned}\Gamma_{\mathrm{rad.}} &= \Gamma_{\mathrm{Vi}}+\Gamma_{\mathrm{Re}} = \Gamma_0\frac{N_{\mathrm{c}}\,|V_{\mathrm{q_2q_3}}|^2\,\alpha_{\mathrm{s}}}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}\int_0^1 \mathrm{d}\xi\,(1-\xi)^2\,[1+2\,\xi]\left[-4+\frac{19}{4}\right]\\ &= N_{\mathrm{c}}\Gamma_0\,|V_{\mathrm{q_2q_3}}|^2\,\frac{\alpha_{\mathrm{s}}}{\pi} = N_{\mathrm{c}}\frac{G_{\mathrm{F}}^2\,M^5}{192\pi^3}\,|V_{\mathrm{q_2q_3}}|^2\,\frac{\alpha_{\mathrm{s}}}{\pi}\,. \end{aligned} \quad (14.83)$$

In the limit of massless quarks when the tree diagram rate $\Gamma_0\,N_{\mathrm{c}}\,|V_{\mathrm{q_2q_3}}|^2$ as given by (13.60) is added to the one-loop QCD corrections (83), then together with $|V_{\mathrm{ud}}|^2+|V_{\mathrm{us}}|^2 \approx 1$, the inclusive semileptonic decay width of the τ lepton is given by

$$\Gamma(\tau \to \nu_\tau + \text{ hadrons }) = N_{\mathrm{c}}\frac{G_{\mathrm{F}}^2\,M^5}{192\pi^3}\left[1+\frac{\alpha_{\mathrm{s}}}{\pi}\right]\,. \quad (14.84)$$

As already mentioned at the beginning, this QCD correction is identical to the correction of the ratio R defined in (13.65) for $\mathrm{e}^+ + \mathrm{e}^-$ annihilation into hadrons. One has the same five diagrams, except that the photon replaces the W weak boson and the vertex γ^μ replaces $\gamma^\mu(1-\gamma_5)$. Thus

$$R = \frac{\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \text{hadrons})}{\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \mu^+ + \mu^-)} = N_{\mathrm{c}}\sum_j Q_j^2\,\left[1+\frac{\alpha_{\mathrm{s}}}{\pi}\right]\,. \quad (14.85)$$

We notice that the factor π^2 in (62) and (82) comes from the second derivative of the $\Gamma(x)$ function [see (63)]. In the sum $\Gamma_{\mathrm{Vi}}+\Gamma_{\mathrm{Re}}$, it happens that the π^2 terms cancel out. However, there are circumstances in which the π^2 term remains. Examples of such cases are the one-loop QED correction to muon or tau lepton decays, as given by (13.27), and the one-loop QCD correction to quark decays $\mathrm{Q} \to \mathrm{q}_1 + \mathrm{q}_2 + \overline{\mathrm{q}}_3$ illustrated by (16.6).

This chapter ends with a remark. When the fermionic masses $m_2 \neq m_3 \neq 0$ are taken into account, calculations of radiative corrections are exceedingly complicated. We simply report the result obtained by replacing in (83) Γ_0 by $\Gamma_0\, G(x_2, x_3)$, and (α_s/π) by $(\alpha_s/\pi)K(x_2, x_3)$ where $x_k = m_k^2/M^2$. Of course $G(0,0) = K(0,0) = 1$. The results in (83) and (84) become

$$\Gamma_{\mathrm{rad.}} = N_{\mathrm{c}}\, |V_{\mathrm{q_2 q_3}}|^2 \left\{ \frac{G_{\mathrm{F}}^2\, M^5}{192\pi^3} G\left(\frac{m_2^2}{M^2}, \frac{m_3^2}{M^2}\right)\right\} \left[\frac{\alpha_{\mathrm{s}}}{\pi} K\left(\frac{m_2^2}{M^2}, \frac{m_3^2}{M^2}\right)\right] \;, \quad (14.86)$$

$$\Gamma(\tau \to \nu_\tau + \text{ hadrons }) = N_{\mathrm{c}} \frac{G_{\mathrm{F}}^2\, M^5}{192\pi^3}\, G\left(\frac{m_2^2}{M^2}, \frac{m_3^2}{M^2}\right) \left[1 + \frac{\alpha_{\mathrm{s}}}{\pi} K\left(\frac{m_2^2}{M^2}, \frac{m_3^2}{M^2}\right)\right] \;.$$

The analytic expression of $G(x, y)$ – corresponding to the tree diagram of Fig. 14.1, uncorrected by QCD – is already given by (13.62). Some numerical values of $K(x, x)$ and $K(x, 0) = K(0, x)$ together with $G(x, x)$ and $G(x, 0) = G(0, x)$ are given in Table 14.1. The decrease of $G(x, y)$ is expected from kinematic phase space effect. What is surprising in the radiative corrections is the spectacular increase of $K(x, y)$ with growing x and y. The mass effect in $K(x, y)$ finds its full application in heavy flavor physics. Its relevance to the decay $\mathrm{b} \to \mathrm{c} + \mathrm{s} + \overline{\mathrm{c}}$ is an example and will be discussed in Chap. 16.

Table 14.1. $G(x,0)$, $G(x,x)$ and $K(x,0)$, $K(x,x)$

$\sqrt{x}$	0	0.1	0.2	0.3	0.4
$G(x,0)$	1	0.93	0.74	0.52	0.32
$G(x,x)$	1	0.85	0.52	0.20	0.026
$K(x,0)$	1	1.62	2.8	4.47	7
$K(x,x)$	1	2.26	4.63	8.15	16.27

Problems

14.1 Noninterference between tree and bremsstrahlung diagrams. To order $\mathcal{O}(g_s^2)$, one can draw two diagrams similar to Fig. 14.3 with two gluons (instead of one) emitted. We call them Fig. 14.3bis. While there is an interference between the diagram of Fig. 14.1 and the loop diagrams of Fig. 14.2 to obtain the $\mathcal{O}(g_s^2)$ corrections to the rate, there is no interference between Fig. 14.1 and Fig. 14.3bis for the same order g_s^2 corrections to the rate. Explain why.

14.2 Analytic expressions of $F_{1,\mathrm{uv}}(q^2)$ and $F_{1,\mathrm{ir}}(q^2)$. For $F_{1,\mathrm{uv}}$ compute the integral in the second line of (17). For $F_{1,\mathrm{ir}}$ in (19), the IR divergence is symbolically written as $\int_0^1 \mathrm{d}u/u$. There are two different ways of parameterizing this IR: either (i) by assigning a fictitious small mass ζ to the gluon, i.e. by replacing its propagator $1/k^2$ in (4) with $1/(k^2-\zeta^2)$; or (ii) by using dimensional regularization as in Sect. 14.4 so that the IR divergence is also represented by a pole $1/\varepsilon$ as the UV divergence. Derive an analytic expression of $F_{1,\mathrm{ir}}(q^2)$ in both cases (i) and (ii).

14.3 $F_1(0)+\delta_\mathrm{q}=0$ from Ward identity. We have seen the trivial role of γ_5 in our QCD corrections to the rate, so let us forget it in the expression of $\Gamma^\mu(p_2,p_3)$ as given by (4). We are considering only the vector current, i.e. QED where photons replace gluons. Multiply $q_\mu=(p_2+p_3)_\mu$ by $\Gamma^\mu(p_2,p_3)$ (without γ_5), and using the expression of $\Sigma(p)$ in (21), show that

$$(p_2+p_3)_\mu\Gamma^\mu(p_2,p_3)=\Sigma(-p_3)-\Sigma(p_2)\ . \tag{14.87}$$

This QED equation, known as the generalized Ward identity, was derived by Takahashi. The original one, pioneered by Ward, can be obtained by letting p_2 tend to $-p_3$. In this limit, we have

$$\Gamma^\mu(p,p)=-\frac{\partial}{\partial p_\mu}\Sigma(p)\ . \tag{14.88}$$

Notice that in (4), if $m_2\neq m_3$, the vector current is not conserved, show that (87) cannot hold. From (12), the left-hand side of (88) is $\Gamma^\mu(p,p)=\gamma^\mu F_1(0)$. Its right-hand side is $-\gamma^\mu\delta_\mathrm{q}$, since $\Sigma(p)=\Sigma(m)+\delta_\mathrm{q}(\not{p}-m)$ from (27) and (31). Then $F_1(0)+\delta_\mathrm{q}=0$. In QED, the counterterm Z_q is usually denoted by $Z_2=1+\delta_2$, so the Ward identity (88) is written as $F_1(0)=1-Z_2$.
On the other hand, the QED counterterm of the vertex denoted by Z_1, which is used to cancel the UV divergence of the form factor $F_1(q^2)$, is given by $F_1(0)=(1/Z_1)-1=1-Z_1+\mathcal{O}(e^4)$, i.e. $F_1(0)=1-Z_1$ (see 15.26). Then together with (88), one has $Z_1=Z_2$.
In brief, the vertex function counterterm $Z_1=1-F_1(0)$ in QED is equal to the fermion field counterterm $Z_2=1+\mathrm{d}\Sigma(p)/\mathrm{d}\not{p}|_{\not{p}=m}$. As we will see in the next chapter, the relation $Z_1=Z_2$ does not hold in QCD.

14.4 $F_2^{\mathrm{em}}(0)$ from Higgs boson contribution. If we replace in Fig. 14.4b, the vertex $\gamma^\mu(1-\gamma_5)$ by γ^μ, and the internal gluon by an internal photon, then we have one-loop QED corrections. We are interested in the finite form factor $F_2^{\mathrm{em}}(0)$ as given by $e^2/(8\pi^2)=\alpha_\mathrm{em}/2\pi$ [see (14)]. The magnetic moment of the electron is usually written as $\frac{1}{2}g\mu_\mathrm{e}$, i.e. $g=2$ corresponds to its pointlike value of $\mu_\mathrm{e}=-e/2m_\mathrm{e}$. Its anomalous magnetic moment, i.e. the deviation from its pointlike value, is $\frac{1}{2}(g-2)=\alpha_\mathrm{em}/2\pi$.
Consider the field $\phi(x)$ of the Higgs boson H which has an interaction with a charged lepton field $\psi(x)$ of mass m_ℓ: $f_\ell\phi(x)\overline{\psi}(x)\psi(x)$. In fact, the coupling

constant is $f_\ell = m_\ell/v$ where $v = (\sqrt{2}G_{\rm F})^{-1/2}$. Similar to Fig. 14.4b, instead of QED corrections, the internal photon field is now replaced by H. Compute the anomalous magnetic moment of the electron $F_2(0)$ due to this virtual H. Experimentally, the deviation $\frac{1}{2}(g-2)$ is known to be 0.0011597 for the electron. What limits on $M_{\rm H}$ can we deduce from this number?

14.5 Fermion mass generated by the gap equation. The Pauli–Villars regularization procedure introduces a large cutoff Λ, i.e. the gluon propagator $1/(k^2-\zeta^2)$ is replaced by

$$\frac{1}{k^2-\zeta^2} \longrightarrow \frac{1}{k^2-\zeta^2} - \frac{1}{k^2-\Lambda^2}\,, \tag{14.89}$$

where ζ is a small gluon mass introduced to regulate the infrared divergence. Compute $\Sigma(p)$ and $B(m^2)$ in terms of Λ. In (25), if the bare mass m_0 is assumed to be zero, then the renormalized mass m in (25) obeys the *gap equation* $B(m^2)=1$. Find m in terms of Λ. This mechanism of mass generation is known as the Nambu–Jona-Lasinio model.

14.6 Mass effect in two-body and three-body phase space. Using (56) in n dimension, first compute $J^{\mu\nu}(q^2)$ as defined by (54) where m_2 and m_3 are not neglected. Compute the three-body phase space

$$\int \frac{\mathrm{d}^3p_1}{2E_1}\frac{\mathrm{d}^3p_2}{2E_2}\frac{\mathrm{d}^3p_3}{2E_3}\delta^4(p_1+p_2+p_3-q) \tag{14.90}$$

as an integral over E_j, E_ℓ where all the masses are $\neq 0$. Give an equivalent of (75) with the three nonzero masses. Show that the E_j, E_ℓ domain of the Dalitz plot is no longer an isosceles triangle, but resembles an ellipse.

Suggestions for Further Reading

Dimensional regularization:

Veltman, M., *Diagrammatica*. Cambridge U. Press, Cambridge 1994

Radiative corrections; mass and field renormalization; infrared effects:

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Weinberg, S., *The Quantum Theory of Fields* (Vol. I). Cambridge U. Press, Cambridge 1995

Phase space integrals in four-dimensions:

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Infrared-safe radiative corrections, using dimensional regularization:

Field, R., *Applications of Perturbative* QCD. Addison-Wesley, Redwood, CA 1989

Guberina, B., Peccei, R. D. and Ruckl, R., Nucl. Phys. **B171** (1980) 333

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Czarnecki, A., Jezabek, M. and Kühn, J. H., Phys. Lett. **346B** (1995) 335

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