13 Muon and Tau Lepton Decays

In this chapter we study the tau lepton $\tau^\pm$ and the muon $\mu^\pm$ decays via weak interactions. As the leptonic numbers are assumed to be strictly conserved in the standard model, the electromagnetic decay modes $\tau^\pm \to \mu^\pm(\mathrm{e}^\pm) + \gamma$ and $\mu^\pm \to \mathrm{e}^\pm + \gamma$ cannot occur. See however Problem 12.1.

The importance of the subject is two-fold. On the one hand, the leptonic decay of τ (or μ) is the simplest and cleanest process which unambiguously determines the left-handed $\mathrm{V} - \mathrm{A}$ structure of the weak charged currents. This left-handed structure is universal in the sense that it describes weak reactions of all particles, whether they are leptons, mesons, or baryons. The violation of discrete symmetries P, C, and CP is well illustrated in the τ leptonic modes. On the other hand, τ is the only lepton massive enough to disintegrate into hadrons. Its semileptonic modes in both exclusive and inclusive channels are ideal for studying the strong interaction in the best possible conditions. The τ semileptonic decays offer an extremely favorable testing ground for both perturbative QCD radiative corrections and non perturbative QCD topics, such as decay constants, form factors, and the conserved vector current (CVC).

To have a global viewpoint and to put the subject in context, we first recall the general framework of weak decays before going into the specific τ and μ cases.

13.1 Weak Decays: Classification and Generalities

As the lightest particles in their categories, the proton and the electron are the only stable charged fermions in nature, a consequence of the conservation of the baryon and lepton numbers (at least to a very high degree of accuracy). On the other hand, when a hadron or charged lepton is produced, it decays more or less quickly into other particles. Hadrons can be grouped into two categories: resonances and low-lying metastable particles. A glance at the some two hundred and fifty existing hadrons in the Review of Particle Physics[1] shows that most of the hadrons are resonances. Examples of resonances are mesons $\rho(770)$, $\mathrm{K}^*(892)$, the charmonia, the bottomonia, as well as the baryons $\Delta(1620)$, $\Sigma_\mathrm{C}(2455)$, etc. Resonances decay quickly by

[1] Phys. Rev. **D54** (1996) 1

strong interactions, for instance $\rho \to 2\,\pi$, $K^* \to K + \pi$, $\Delta \to N + \pi + \pi$; their lifetimes are very short $\sim 10^{-23}$ s. Other neutral particles like the π^0, η, Σ^0 have smaller total widths (or longer lifetimes $\sim 10^{-19}$ s) because they only decay by electromagnetic interactions which violate isospin or G-parity (Chap. 6): $\pi^0 \to \gamma + \gamma$, $\Sigma^0 \to \Lambda + \gamma$, $\eta \to 3\,\pi$ (the G-parity of η is $+1$, whereas an odd number of pions have G-parity -1).

Generally, strong decays conserve quantum numbers such as isospin, flavors, and discrete C, P, T symmetries. Electromagnetic decays only violate isospin, whereas in weak interactions, isospin, flavors and, discrete symmetries are violated.

The weak interaction governs the decay of low-lying metastable hadrons: $\pi^{\pm}$, charged and neutral flavored mesons K, D, D_s, B, B_s^0 as well as baryons n, Λ, $\Sigma^{\pm}$, Ξ, Ω^-, Λ_c^+, Λ_b^0. Their widths are much smaller than those of the resonances, their lifetimes range from 10^3 s for the free neutron to 10^{-13} s for charmed or bottom mesons. Weak decays of leptons, mesons, and baryons appear at first glance to have little in common. Lepton decays involve either only leptons, for example $\mu^- \to \nu_\mu + e^- + \overline{\nu}_e$, or a neutrino and hadrons, e.g. $\tau^- \to \nu_\tau + \pi^- + \pi^0$ (semileptonic modes). Mesons can decay into only lepton pairs, for instance $K^+ \to \mu^+ + \nu_\mu$, or into hadrons and a pair of leptons (semileptonic) such as $D^0 \to \rho^- + e^+ + \nu_e$, or into pure hadrons, e.g. $B \to J/\psi + K^*$. Baryons can have both semileptonic and purely hadronic channels. Isospin, parity, and charge conjugation are violated in weak decays. In most cases, hadronic flavors (strangeness, charm, bottomness) change. Because of the conservation rules mentioned above, strong and electromagnetic interactions are forbidden in the decays of low-lying metastable hadrons, otherwise weak interactions are swamped by them.

To lowest order of G_F, the neutral gauge boson Z does not participate in flavor-changing weak decays because of the GIM mechanism. Flavor-conserving weak decays by neutral currents are many orders of magnitude smaller than electromagnetic decays (Chap. 12). Therefore the Z-mediated weak decays will not be considered, only charged current processes are studied. It is remarkable that weak decays of particles, in spite of the huge differences in the various channels, partial rates, and lifetimes, all share a common feature. They can be quantitatively described by an *effective Lagrangian* which is the product of two left-handed V − A charged currents mediated by the gauge bosons W, as mentioned in (9.1)–(9.5):

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathrm{i} \lim_{q^2 \ll M_W^2} \left(\frac{-\mathrm{i}g}{2\sqrt{2}}\right)^2 \left(L^\lambda + H^\lambda\right) \frac{-\mathrm{i}g_{\lambda\rho}}{q^2 - M_W^2} \left(L^\rho + H^\rho\right)^\dagger \\ &= \frac{G_F}{\sqrt{2}} \left(L^\lambda + H^\lambda\right)\left(L_\lambda^\dagger + H_\lambda^\dagger\right) \ , \end{aligned} \tag{13.1}$$

where L^λ and H^λ are the leptonic and hadronic currents, with the Fermi coupling constant G_F defined by $G_F/\sqrt{2} = g^2/8M_W^2$. Weak decays of all particles, in particular the modes in Table 6.6, are described by (1):

- leptonic decays of leptons by $L^\lambda L^\dagger_\lambda$,
- leptonic modes of mesons and semileptonic decays by $L^\lambda H^\dagger_\lambda + H^\lambda L^\dagger_\lambda$,
- nonleptonic or hadronic decays of hadrons by $H^\lambda H^\dagger_\lambda$.

These V − A charged currents are expressed in terms of lepton and quark fields, they may be written as

$$L_\lambda = \sum_{\ell=\mathrm{e},\mu,\tau} \overline{\nu}_\ell(x)\gamma_\lambda(1-\gamma_5)\ell(x) \ ,$$

$$H_\lambda = \sum_{\mathrm{Q}\,,\,\mathrm{q}} V_{\mathrm{Q\,q}} H^{Qq}_\lambda \ , \quad \text{where } \ H^{Qq}_\lambda = \overline{Q}(x)\gamma_\lambda(1-\gamma_5)q(x) \ , \tag{13.2}$$

where $Q = u$, c and $q = d$, s , b fields. In (2), V_{Qq} is a CKM matrix element, Q stands for the up and charmed quark fields, while q represents the down, strange, and bottom quark fields. The reason for the absence of the top quark is that top is heavier than the gauge boson W and can decay directly into $\mathrm{t} \to \mathrm{W} + \mathrm{b}$ with the coupling $g \sim M_{\mathrm{W}}\sqrt{G_{\mathrm{F}}}$ without passing through the virtual W propagator as in (1).

The universal weak effective Lagrangian (1) has a long history, starting with the neutron and muon decays together with the crucial discovery of parity violation in 1956. This Lagrangian is now the core of the standard electroweak theory. The τ decay offers a powerful test of (1) in both leptonic and semileptonic channels. Moreover in semileptonic modes, the interplay between QCD and weak interactions can be fully exploited.

The Feynman diagrams for the μ and τ decays are drawn in Fig. 13.1. At the quark level, only the doublet (u, d″) enters, where $\mathrm{d}'' = V_{\mathrm{ud}}\ \mathrm{d} + V_{\mathrm{us}}\ \mathrm{s}$. Since the τ is lighter than the charm, the other doublet (c, s″) where $\mathrm{s}'' = V_{\mathrm{cs}}\ \mathrm{s} + V_{\mathrm{cd}}\ \mathrm{d}$ does not intervene. The τ decay products are light mesons formed by u, d, and s quarks, such as the unflavored π, ρ, and $\mathrm{a}_1(1260)$ associated with $|V_{\mathrm{ud}}|^2 \approx (0.97)^2$ for the Cabibbo-favored modes, and the strange K, K*(892) with $|V_{\mathrm{us}}|^2 \approx (0.22)^2$ for the Cabibbo-suppressed modes.

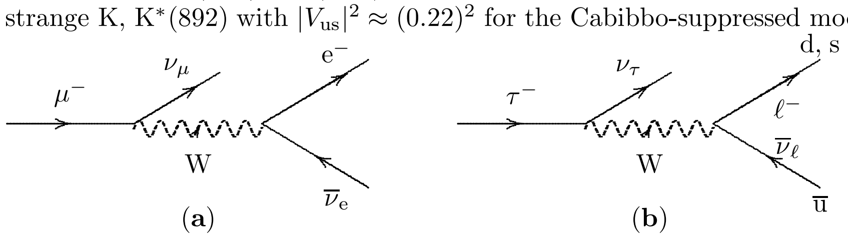


Fig. 13.1. (**a**) The only possible decay mode of the light μ^- ; (**b**) leptonic and semileptonic decays of the heavy τ^-

Let us close the section with one remark. In any reaction, the branching ratio of a parent particle P decaying into any particular channel F is the first quantity to be measured:

$$\mathrm{B}_F \equiv \frac{\Gamma(P \to F)}{\Gamma_{\mathrm{total}}(P)} = \tau_P \times \Gamma(P \to F) \ , \ \ (\tau_P \text{ is the lifetime of } P) \ .$$

Important weak decay dynamics can be revealed by the branching ratios of processes into different final states.

13.2 Leptonic Modes

Let us start with the important leptonic decay mode of τ:

$$\tau^-(P) \to \nu_\tau(k_1) + \ell^-(p) + \overline{\nu}_\ell(k_2) \ , \tag{13.3}$$

where ℓ^- stands for the electron or the muon; the four-momentum of these particles are specified in parentheses. The τ^- and the ℓ^- masses are denoted by M and m respectively. The neutrinos are assumed massless. This mode is decisive for the discovery of τ. Produced in $\mathrm{e}^+ + \mathrm{e}^- \to \tau^+ + \tau^-$, the τ^+ and τ^- leptons subsequently decay with a distinctive signature rarely found in other particle decays. Indeed, from $\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$ and $\tau^+ \to \overline{\nu}_\tau + \mu^+ + \nu_\mu$, one observes in the decay product a pair $\mathrm{e}^-\mu^+$ + invisible neutrinos whose presence is revealed by the apparent missing energy. It was through this special signature $\mathrm{e}^\mp\mu^\pm$ that the $\tau^\pm$ leptons were discovered.

13.2.1 Leptonic Branching Ratio

We first note that a very naive estimate of the leptonic branching ratio $\mathrm{Br}(\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e}) \equiv \mathrm{B_e}$ can be made by a simple counting rule. Indeed the τ has two pure leptonic modes leading to emission of electrons and muons. It also has the semileptonic modes, its *inclusive decay* defined as $\tau \to \nu_\tau +$ any hadron is symbolically written as $\tau \to \nu_\tau + X$, where X stands for the sum of all kinematically allowed mesons. This inclusive semileptonic process may be described by τ decays into its own neutrino ν_τ and a quark+antiquark pair. In the parton model spirit, this approach is justified by a large energy released by the τ. From the closure argument, these quark pairs saturate the sum of all the hadronic modes, since once quarks are produced by a weak decay, they can only form hadrons. The inclusive semileptonic rate is given by

$$\Gamma(\tau^- \to \nu_\tau + X) = \Gamma(\tau^- \to \nu_\tau + \mathrm{d}_j + \overline{\mathrm{u}}_j) + \Gamma(\tau^- \to \nu_\tau + \mathrm{s}_j + \overline{\mathrm{u}}_j) \, ,$$

where j is the color index. The first (second) term on the right-hand side of the above equation is the decay rate into kinematically allowed nonstrange (strange) mesons. They are respectively associated with $|V_\mathrm{ud}|^2 \approx (0.97)^2$ and $|V_\mathrm{us}|^2 \approx (0.22)^2$. As we will see later in (22) and (62), the rate depends on the fermionic masses in the final state. However, in the first approximation we may neglect their masses, so that for each color j of quarks, we have

$$\Gamma(\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e}) \approx \Gamma(\tau^- \to \nu_\tau + \mathrm{q}_j + \overline{\mathrm{u}}_j) \ \ , \ \text{where} \ \ \mathrm{q} = \mathrm{d}, \mathrm{s} \ . \tag{13.4}$$

Since quarks have $N_\mathrm{c} = 3$ colors and $|V_\mathrm{ud}|^2 + |V_\mathrm{us}|^2 \approx 1$, the inclusive semileptonic width $\Gamma(\tau^- \to \nu_\tau + X)$ is three times $\Gamma(\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e})$, and the total width is five times the latter (do not forget the muon), so the leptonic branching ratio $\mathrm{B_e}$ is $\frac{1}{5}$ by this counting rule to be compared with the experimental data of $(17.83 \pm 0.06)\%$. The difference can be explained by the mass and QCD correction effects at the quark level which amount to about 10%. This simple counting rule supports the quark–parton picture which in turn indicates that the energy released by the τ is large enough to make legitimate the use of the parton model. This color-counting argument holds also in high energy $\mathrm{e}^+ + \mathrm{e}^-$ annihilation into hadrons. The similarity of $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$ and $\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \text{hadrons})$ may be seen in the following relation (where W^* and γ^* are the virtual W boson and photon):

$$\begin{aligned}&\tau^- \to \nu_\tau + X = \tau^- \to \nu_\tau + \mathrm{W}^* \ , \ \text{followed by } \mathrm{W}^* \to \mathrm{q}_j\,\overline{\mathrm{u}}_j \Rightarrow \text{hadrons} \ ,\\ &\mathrm{e}^+ + \mathrm{e}^- \to \gamma^* \ , \ \text{followed by } \gamma^* \to \mathrm{Q_j}\,\overline{\mathrm{Q}}_\mathrm{j} \ \Rightarrow \text{hadrons} \ . \end{aligned} \tag{13.5}$$

13.2.2 Parity Violation. Energy Spectrum

We go further by computing the ℓ^- angular distribution and its asymmetry with respect to the τ polarization axis, the ℓ^- energy spectrum, and finally the integrated leptonic width $\Gamma(\tau^- \to \nu_\tau + \ell^- + \overline{\nu}_\ell)$. All of these physical quantities are of great importance in the determination of the τ properties, in particular its weak coupling strength and the structure of its charged current. Their measurements can give a definite answer to the question: is the τ lepton a replica of the electron and the muon, or is there any deviation from the standard model? To allow for possible deviations from the pure left-handed V − A current of the τ–ν_τ system, let us write the (current × current) decay amplitude of (3) in a more general form:

$$\mathcal{M} = \frac{G_\mathrm{F}}{\sqrt{2}} \ \{\overline{u}(k_1)\gamma_\lambda(a - b\gamma_5)u(P)\} \ \big\{\overline{u}(p)\gamma^\lambda(1-\gamma_5)v(k_2)\big\} \ . \tag{13.6}$$

A V − A current of the τ–ν_τ system corresponds to $a = b$, and in the standard electroweak model $a = b = 1$ (universality of the three lepton families). This property is well established for the e–ν_e and μ–ν_μ systems, as explicitly shown by the second factor $\big\{\overline{u}(p)\gamma^\lambda(1-\gamma_5)v(k_2)\big\}$ in (6). A V + A structure of the τ–ν_τ current corresponds to $a = -b$. Arbitrary a and b correspond to a mixture of left-handed and right-handed currents. We first evaluate $|\mathcal{M}|^2 = \frac{1}{2}\,G_\mathrm{F}^2\,(T_1)_{\lambda\rho}(T_2)^{\lambda\rho}$, where

$$\begin{aligned}(T_1)_{\lambda\rho} &= \mathrm{Tr}\big[u(k_1)\overline{u}(k_1)\gamma_\lambda(a-b\gamma_5)u(P)\overline{u}(P)\gamma_\rho(a-b\gamma_5)\big] \ ,\\ (T_2)^{\lambda\rho} &= \mathrm{Tr}\big[u(p)\overline{u}(p)\gamma^\lambda(1-\gamma_5)v(k_2)\overline{v}(k_2)\gamma^\rho(1-\gamma_5)\big] \ .\end{aligned} \tag{13.7}$$

In order to study the angular distribution of ℓ^- with respect to the τ spin, we sum only the spins of the final state but still keep untouched S^μ_τ, the spin polarization of the initial state τ. In the τ rest frame $P^\mu = (M, \mathbf{0})$, its spin vector S_τ^μ takes the form $S_\tau^\mu = (0, \widehat{\boldsymbol{S}})$, with $|\widehat{\boldsymbol{S}}| = 1$. Recalling that $u(P)\overline{u}(P) = \frac{1}{2}(\not{P} + M)[1 + \gamma_5 \not{S}_\tau]$, we find

$$\begin{aligned}
(T_1)_{\lambda\rho} &= \Big(\frac{a+b}{2}\Big)^2 \mathrm{Tr}\Big[\not{k}_1\gamma_\lambda(\not{P} - M\not{S}_\tau)\gamma_\rho(1-\gamma_5)\Big] \\
&\quad + \Big(\frac{a-b}{2}\Big)^2 \mathrm{Tr}\Big[\not{k}_1\gamma_\lambda(\not{P} + M\not{S}_\tau)\gamma_\rho(1+\gamma_5)\Big] , \\
(T_2)^{\lambda\rho} &= 2\ \mathrm{Tr}\Big[\not{p}\gamma^\lambda \not{k}_2\gamma^\rho(1-\gamma_5)\Big] . \qquad (13.8)
\end{aligned}$$

The relation (12.39) is useful to distinguish the effect of (V ∓ A) × (V ∓ A) product of currents from the (V ± A) × (V ∓ A) one. We get

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}|^2 &= 64\, G_\mathrm{F}^2 \Big\{ \Big(\frac{a+b}{2}\Big)^2 \big[p\cdot k_1\big]\, \big[(P - MS_\tau)\cdot k_2\big] \\
&\quad + \Big(\frac{a-b}{2}\Big)^2 \big[k_1\cdot k_2\big]\, \big[(P + MS_\tau)\cdot p\big]\Big\} . \qquad (13.9)
\end{aligned}$$

The general formula for the computation of decay widths is given by (4.70). In our case, with three particles in the final state, we have

$$\begin{aligned}
\mathrm{d}\Gamma &= \frac{1}{2M}\frac{\mathrm{d}^3p}{2E}\int\!\!\int \frac{\mathrm{d}^3k_1}{2E_1}\,\frac{\mathrm{d}^3k_2}{2E_2}\,\frac{\delta^4(k_1+k_2-q)}{(2\pi)^5}\sum_{\text{spins}}|\mathcal{M}|^2, \quad q = P - p \\
&= \frac{64\, G_\mathrm{F}^2}{2M}\frac{\mathrm{d}^3p}{2E}\Big\{\Big(\frac{a+b}{2}\Big)^2 p_\mu\,(P - MS_\tau)_\nu + \Big(\frac{a-b}{2}\Big)^2 (P + MS_\tau)\cdot p\; g_{\mu\nu}\Big\} \\
&\quad \times \frac{1}{(2\pi)^5}\int\!\!\int \frac{\mathrm{d}^3k_1}{2E_1}\,\frac{\mathrm{d}^3k_2}{2E_2}\delta^4(k_1+k_2-q)\; k_1^\mu\, k_2^\nu , \qquad (13.10)
\end{aligned}$$

there is no factor $\frac{1}{2}$ in $\sum_{\text{spins}}|\mathcal{M}|^2$ on the right-hand side of (10) because the τ spins are not averaged. Since the neutrinos are unobserved, we first integrate over their three-momenta $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. On the other hand, to study the energy and angular distributions of ℓ^-, we keep its momentum $\boldsymbol{p}$ untouched at the beginning. The last term on the right-hand side of (10) can be computed using formulas in the Appendix. With massless neutrinos, the integration is simple (Problem 5.2) and we get

$$I^{\mu\nu} \equiv \int \frac{\mathrm{d}^3k_1}{2E_1}\,\frac{\mathrm{d}^3k_2}{2E_2}\delta^4(k_1+k_2-q)\; k_1^\mu\, k_2^\nu = \frac{\pi}{24}\big(q^2 g^{\mu\nu} + 2q^\mu q^\nu\big) . \qquad (13.11)$$

The product of $I^{\mu\nu}$ with the quantity in the curly brackets of (10) is

$$\begin{aligned}
&\frac{\pi}{24}\Big\{\Big(\frac{a+b}{2}\Big)^2\Big[q^2[(P - MS_\tau)\cdot p] + 2\,p\cdot q\,[(P - MS_\tau)\cdot q]\Big] \\
&\quad + \Big(\frac{a-b}{2}\Big)^2 6\,q^2\,[(P + MS_\tau)\cdot p]\Big\} . \qquad (13.12)
\end{aligned}$$

To distinguish the effects of the V − A current from those of a possible V + A current, it is convenient to arrange (12) into two parts. The first part is a sum of left and right chiral current contributions *with equal weights*, and the second part belongs to their unequal mixture. This separation enables us to introduce later the Michel parameters ρ, ξ, and δ, which are important measurable quantities to test whether or not the heavy lepton τ is a replica of the muon and the electron. This decomposition turns out to be a powerful method of investigating the decay dynamics, as we will see. With the coefficient $\pi/24$ implicitly understood, let us rewrite (12) in a form in which the mentioned separation is explicit [note that $S_\tau \cdot q \equiv S_\tau \cdot (P-p) = -S_\tau \cdot p$]. The quantity inside the curly brackets of (12) is

$$\begin{aligned}
&6\,q^2\,P\cdot p\Big[\Big(\frac{a+b}{2}\Big)^2+\Big(\frac{a-b}{2}\Big)^2\Big] + \Big[2\,p\cdot q\,P\cdot q - 5\,q^2\,P\cdot p\Big]\Big(\frac{a+b}{2}\Big)^2 \\
&+ MS_\tau\cdot p\Big\{2\,q^2\,\Big[\Big(\frac{a+b}{2}\Big)^2+\Big(\frac{a-b}{2}\Big)^2\Big] + (2\,p\cdot q - 3\,q^2)\Big(\frac{a+b}{2}\Big)^2 \\
&+ 4\,q^2\Big(\frac{a-b}{2}\Big)^2\Big\}\ . \qquad (13.13)
\end{aligned}$$

In (13), this separation applies to both the spin-dependent $MS_\tau \cdot p$ and the spin-independent $q^2\,P\cdot p$ terms. In the τ rest frame, θ denotes the angle between the three-momentum $\boldsymbol{p}$ of the ℓ^- and the spin $\widehat{\boldsymbol{S}}$ of the τ^-, thus

$$\begin{aligned}
&p^\mu = (E,\boldsymbol{p})\ ,\ S_\tau\cdot p = -|\boldsymbol{p}|\cos\theta\ ,\ \mathrm{d}^3p = 2\pi\,\mathrm{d}(\cos\theta)\,|\boldsymbol{p}|\,E\,\mathrm{d}E\ , \\
&q^2 = M^2 - 2ME + m^2\ \ .\ \text{Since } q^2 \ge 0 \Rightarrow m \le E \le \frac{M^2+m^2}{2M} \equiv E_{\max}\ , \\
&P\cdot p = ME\ \ ,\ \ P\cdot q = M(M-E)\ \ ,\ \ p\cdot q = ME - m^2\ \ . \qquad (13.14)
\end{aligned}$$

Putting (10), (13), and (14) together, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta\,\mathrm{d}E} = \frac{G_\mathrm{F}^2\,|\boldsymbol{p}|\,E}{3(2\pi)^3}\ \frac{a^2+b^2}{2}\big\{X\big\}\ ,\ \ \text{where} \qquad (13.15) \\
&\big\{X\big\} = \Big\{6\,q^2 + \frac{(a+b)^2}{2(a^2+b^2)}\,\big[8ME - 3M^2 - m^2[3+(2M/E)]\big] \\
&-\ \frac{|\boldsymbol{p}|}{E}\cos\theta\Big[2\,q^2\,\Big(1+\frac{(a-b)^2}{a^2+b^2}\Big) + \frac{(a+b)^2}{2(a^2+b^2)}\,(8ME - 3M^2 - 5m^2)\Big]\Big\}.
\end{aligned}$$

In the above expression of $\{X\}$, the $\cos\theta$-independent term (isotropic part) gives the energy spectrum of the emitted ℓ^-. The $\cos\theta$ term (anisotropic part) on the last line represents the angular correlation between the spin $\boldsymbol{S}$ of the decaying particle and the three-momentum $\boldsymbol{p}$ of the outgoing ℓ^-. This correlation is a crucial quantity to reveal the parity violation of weak interaction. A short recall of the discussions given in Chap. 5 might be useful.

Parity Violation. The parity violation phenomenon can only appear as a *pseudoscalar term* constructed from experimentally measurable quantities, for which the anisotropic part of (15) is the simplest example. This apparently trivial fact had never been noticed before 1956, when Lee and Yang pointed out that if one did not look for a pseudoscalar measurable quantity, the non-conservation of parity could never be experimentally discovered, even if the interaction violates space inversion (or parity) symmetry P.

In the current $\times$ current amplitude, the interference $\mathrm{V} \times \mathrm{A}$ is a pseudoscalar quantity. The electron energy spectrum and the integrated rate are two examples of scalar quantities coming from the $\mathrm{V} \times \mathrm{V}$ and $\mathrm{A} \times \mathrm{A}$ products of two curents. Their measurements cannot tell whether the P symmetry is broken by weak interactions or not.

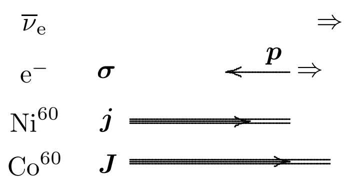


Fig. 13.2. $\boldsymbol{J} \cdot \boldsymbol{p}$ correlation in $\mathrm{Co}^{60} \to \mathrm{Ni}^{60} + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$

As explained in Chap. 5, under space inversion P, $\boldsymbol{x} \to -\boldsymbol{x}$, $\boldsymbol{S} \to +\boldsymbol{S}$ while $\boldsymbol{p} \to -\boldsymbol{p}$, the non conservation of parity manifests itself by a nonzero value of the coefficient of the pseudoscalar term $\boldsymbol{S} \cdot \boldsymbol{p} = |\boldsymbol{p}| \cos\theta$ which occurs in the matrix element of the operator product $\mathrm{V} \times \mathrm{A}$. A forward–backward asymmetry in the emission of l with respect to the spin $\boldsymbol{S}$ constitutes an unequivocal proof of parity violation. Very similar to the $\boldsymbol{S} \cdot \boldsymbol{p}$ correlation considered here is the electron asymmetry with respect to the polarization axis of the cobalt nucleus in $\mathrm{Co}^{60} \to \mathrm{Ni}^{60} + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$ observed by C. S. Wu, who gave the first experimental demonstration of parity violation. A sample of Co^{60} was kept at a very low temperature, its spin $\boldsymbol{J}$ ($J = 5$) is aligned and the final Ni^{60} has spin $j = 4$. The electron angular distribution is described by the function $\mathcal{F}(\theta) = 1 + \alpha\, \boldsymbol{J} \cdot \boldsymbol{p} / |\boldsymbol{J}|\, E$, where $\boldsymbol{p}$ and E are the momentum and energy of the electron. If the coefficient α is found to be definitely nonzero, a parity violation is proven. This was indeed the case, and the electron was found to be emitted preferentially antiparallel to $\boldsymbol{J}$, i.e. $\alpha = -1$. The difference by one unit of spin between the initial and final nuclei on the one hand, and the conservation of the z component of the angular momentum along $\boldsymbol{J}$ on the other hand, imply that the electron spin $\boldsymbol{\sigma}$ must point in the direction $\boldsymbol{J}$. It shows that the electron emitted in nuclear β-decay is antiparallel to its spin $\boldsymbol{\sigma}$, i.e. the electron is left-handed. As illustrated in Fig. 13.2, this is the first experimental indication of the $\mathrm{V} - \mathrm{A}$ character of the charged current. On the other hand, scalar quantities, such as the isotropic energy spectrum in (15), have no bearing on the question of parity violation and cannot be used to test it.

Energy Spectrum. Integrating (15) over θ, we obtain the energy spectrum of ℓ^-. The anisotropic term, which is linear in $\cos\theta$, vanishes, while the isotropic term is doubled. We have

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E} = \frac{a^2+b^2}{2}\frac{G_\mathrm{F}^2\,|\boldsymbol{p}|\,E}{3(2\pi)^3}\Big\{12\,(M^2-2ME+m^2) + \frac{8}{3}\rho\Big[8ME-3M^2-m^2[3+(2M/E)]\Big]\Big\}\ , \tag{13.16}$$

where we define the Michel parameter ρ by

$$\rho = \frac{3}{8}\,\frac{(a+b)^2}{a^2+b^2}\ . \tag{13.17}$$

For historic reasons (see below), conventionally, we write ρ with the coefficients 8/3 in (16) or 3/8 in (17), since ρ turns out to be 3/4 in the four-fermion interaction of the type $(\mathrm{V}-\mathrm{A})\times(\mathrm{V}-\mathrm{A})$ product of currents. We now see that the ℓ^- energy spectrum $\mathrm{d}\Gamma/\mathrm{d}E$ in τ decay (Fig. 13.3) is very useful because it distinguishes the $\mathrm{V}\pm\mathrm{A}$ property of the τ-ν_τ weak current. Although the shape $\mathrm{d}\Gamma/\mathrm{d}E$ for $\mathrm{V}-\mathrm{A}$ is distinct from the $\mathrm{V}+\mathrm{A}$ one, their integrated rates $\Gamma = \int \frac{\mathrm{d}\Gamma}{\mathrm{d}E}\mathrm{d}E$ are identical. Hence measurement of Γ alone cannot distinguish the chirality of weak currents.

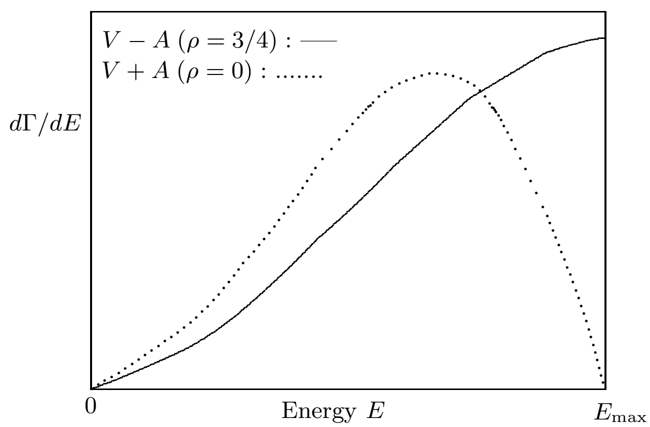


Fig. 13.3. The electron energy spectrum $\mathrm{d}\Gamma/\mathrm{d}E$ in $\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$.

In the 1950s, the theory of weak interaction was still in an embryonic state, and it was not known whether the four-fermion β-decay of nuclei or muon was of the current $\times$ current form, e.g. $\mathrm{V}\times\mathrm{V}$ or $[\overline{\psi}_1\gamma_\mu\psi_2]\ [\overline{\psi}_3\gamma^\mu\psi_4]$, as postulated by Fermi in analogy with the electromagnetic interaction. At that time, without data on the parity violation, the weak interaction could be *a priori* any combination of the scalars obtained from the covariant products $\mathrm{S}\times\mathrm{S}$, $\mathrm{V}\times\mathrm{V}$, $\mathrm{T}\times\mathrm{T}$, $\mathrm{A}\times\mathrm{A}$, $\mathrm{P}\times\mathrm{P}$ (Chap. 5). Michel's idea is to introduce in the electron energy spectrum a parameter ρ to separate terms which are *common* to the structures S, V, T, A, P from other terms which are *sensitive* to some of these structures. For example, a pure product $\mathrm{S} \times \mathrm{S}$ would give $\rho = 0$, while a pure $\mathrm{T} \times \mathrm{T}$ would give $\rho = 1$.

Following Michel, we are led to rearrange (16) into two terms separated by a parameter ρ. The first term $12\,(M^2 - 2ME + m^2)$, which is independent of a and b, cannot distinguish the $\mathrm{V}-\mathrm{A}$ from the $\mathrm{V}+\mathrm{A}$ structure of the τ-ν_τ current. The second term [last line of (16)], which depends on a and b, is sensitive to $\mathrm{V} \mp \mathrm{A}$ and can be used to determine this $\mathrm{V} \mp \mathrm{A}$ structure. We emphasize that in the decay $\tau^- \to \nu_\tau + \ell^- + \overline{\nu}_\ell$, once the current of the final state ℓ–ν_ℓ is known to have the $\mathrm{V}-\mathrm{A}$ structure, then the $\mathrm{V} \pm \mathrm{A}$ current of the initial state τ–ν_τ can be determined by measuring the parameter ρ. We extract ρ by fitting the electron energy distribution (16) with data and obtain important information on the dynamics.

From (17), ρ is always $\leq \frac{3}{4}$. The $\mathrm{V}-\mathrm{A}$ ($a = b$) of the τ–ν_τ current in (6) corresponds to $\rho = \frac{3}{4}$, a pure V ($b = 0$) or a pure A ($a = 0$) would result in $\rho = \frac{3}{8}$, while a $\mathrm{V}+\mathrm{A}$ type ($a = -b$) would imply $\rho = 0$. Recent data[2] give $\rho_\tau = 0.742 \pm 0.027$, in excellent agreement with the $\mathrm{V}-\mathrm{A}$ charged current involved in τ decays. For the muon, the ρ_μ parameter measured in muon decay $\mu^- \to \nu_\mu + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$ is 0.7518 ± 0.0026.

13.2.3 Angular Distribution. Decay Rate

The confirmation of $a = b$ is again found in the $\boldsymbol{p} \cdot \boldsymbol{S}$ correlation. We rewrite the anisotropic part of (15) in terms of two additional Michel parameters usually denoted as ξ and δ:

$$\begin{aligned}\{X\} &= 6\,(M^2 - 2ME + m^2) + \frac{4}{3}\rho\left[8ME - 3M^2 - m^2\left(3 + \frac{2M}{E}\right)\right] \\ &- \xi\,\frac{|\boldsymbol{p}|}{E}\cos\theta\Big[2\,(M^2 - 2ME + m^2) + \frac{4}{3}\,\delta\,(8ME - 3M^2 - 5m^2)\Big]\,, \end{aligned} \tag{13.18}$$

$$\xi = 1 + \frac{(a-b)^2}{a^2 + b^2} \quad , \quad \delta = \frac{3}{8}\,\frac{(a+b)^2}{a^2 + b^2 + (a-b)^2}\,. \tag{13.19}$$

For fixed E, the parameters ξ and δ can be obtained by fitting the θ distribution with experiments. Data[2] give $\xi = 1.03 \pm 0.12$, and $\xi\,\delta = 0.76 \pm 0.11$; they are in excellent agreement with $a = b$. For a $\mathrm{V}+\mathrm{A}$ current, we would get $\xi = 3$, $\delta = 0$, in sharp contrast with $\xi = 1$, $\delta = \frac{3}{4}$ for a $\mathrm{V}-\mathrm{A}$.

Equivalently, for fixed E, the angular distribution of the ℓ^- with respect to the spin direction of the τ^- in (15) can be written as $D(\theta)$. With an overall normalization factor not explicitly shown and neglecting m^2 in (15), $D(\theta)$ can be written as

$$\begin{aligned} D(\theta) = 1 - \alpha\cos\theta \ , \ \text{ where } \ &\alpha = \frac{4E - M}{3M - 4E} \quad \text{for } \mathrm{V}-\mathrm{A} \\ &\alpha = +1 \qquad\qquad \text{for } \mathrm{V}+\mathrm{A}\,. \end{aligned} \tag{13.20}$$

[2] Phys. Rev. **D54** (1996) 1

The forward–backward asymmetry is also an important physical measurable quantity that can fix the relative sign between a and b; its measurement is therefore useful. For a $\mathrm{V}-\mathrm{A}$ current, the asymmetry parameter α increases from $-\frac{1}{3}$ at $E=0$ to $+1$ when E reaches $E_{\max}=M/2$, whereas α is constant in the $\mathrm{V}+\mathrm{A}$ case.

From now on, we put $a=b$ but still let the overall normalization factor a be arbitrary. We now see that data fix a to 1, i.e. the *universality* of e, μ, τ is experimentally confirmed.

Putting $\rho=\frac{3}{4}$ and $a=b$, we now integrate (16) to obtain the full leptonic rate; the integration range for E is given in (14):

$$\Gamma=\frac{2\,a^2\,G_{\mathrm{F}}^2}{3\,(2\pi)^3}\int_m^{E_{\max}}\sqrt{E^2-m^2}\;\left[ME\,(3M-4E)-m^2\,(2M-3E)\right]\,\mathrm{d}E\;.$$

The result of the above integration is

$$\Gamma(\tau^-\to\nu_\tau+\ell^-+\overline{\nu}_\ell)=a^2 f(m^2/M^2)\,\Gamma_0\;\;,\quad\Gamma_0\equiv\frac{G_{\mathrm{F}}^2M^5}{192\,\pi^3}\;,\qquad(13.21)$$

$$f(x)=1-8\,x+8\,x^3-x^4-12\,x^2\,\log x\quad;\quad f(m_\mu^2/M^2)=0.9728\;.(13.22)$$

The formula $\Gamma_0\equiv G_{\mathrm{F}}^2M^5/192\,\pi^3$ – which gives the decay rate of a fermion of mass M into three massless fermions (Problem 5.2) – will be repeatedly used. The phase space correction $f(m^2/M^2)$ takes into account one massive fermion among the three in the final state, the other two are massless. The numerical value of $\Gamma(\tau^-\to\nu_\tau+\mathrm{e}^-+\overline{\nu}_{\mathrm{e}})$ can be obtained using data of both the τ lifetime and the electronic branching ratio

$$\Gamma(\tau^-\to\nu_\tau+\mathrm{e}^-+\overline{\nu}_{\mathrm{e}})=\frac{\mathrm{B_e}}{\tau_\tau}=\frac{0.1783\pm0.0006}{2.91\pm0.015}\times10^{13}\;\mathrm{s}^{-1}\;.\qquad(13.23)$$

To determine the coefficient a, we compare (23) with the theoretical rate (21), in which the phase space correction due to the electron mass m_{e} is completely negligible, i.e. $f(m_{\mathrm{e}}^2/M^2)$ is taken as 1. However the radiative correction 0.996 given below in (28) is included, so that

$$\frac{G_{\mathrm{F}}^2M^5}{192\,\pi^3}\times0.996=4.033\times10^{-13}\;\mathrm{GeV}\;=0.06127\times10^{13}\,\mathrm{s}^{-1}\;\Rightarrow a=1\pm0.006\;.$$

Having shown that $a=b=1$, we rewrite the previous formula:

$$\begin{aligned}\frac{\mathrm{d}\Gamma(\tau^\mp\to\ell^\mp+\nu+\overline{\nu})}{\mathrm{d}\cos\theta\,\mathrm{d}E}=&\frac{G_{\mathrm{F}}^2\,|\boldsymbol{p}|}{24\,\pi^3}\Big\{ME\,(3M-4E)\;-m^2\,(2M-3E)\\&\mp\;|\boldsymbol{p}|\cos\theta\,[M\,(4E-M)-3m^2]\Big\}\;.\end{aligned}\qquad(13.24)$$

The angular distribution for τ^+ can be obtained from the angular distribution of the τ^- by $u(P) \leftrightarrow \overline{v}(P)$, implying $MS_\tau \leftrightarrow -MS_\tau$ in (9), from which (24) follows.

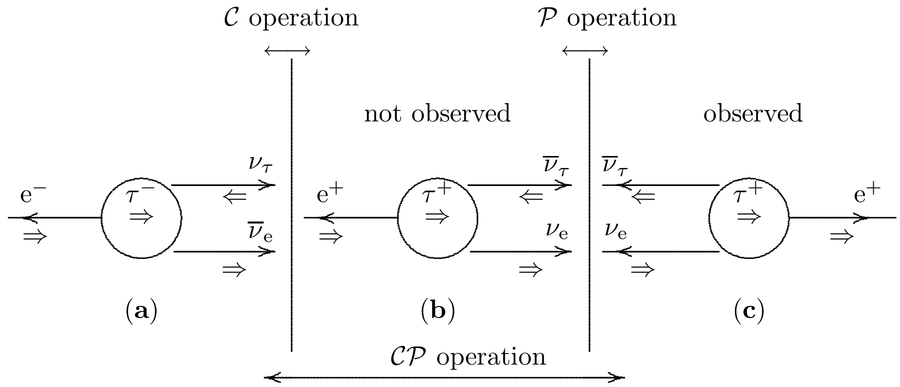


Fig. 13.4. (**a**) At $E = E_{\max}$, the angular distribution of the e^- with respect to the τ^- polarization; (**b**) the charge-conjugation states of (**a**); (**c**) $\mathcal{CP}$ operated on (**a**) gives the angular distribution of the e^+ with respect to the τ^+ polarization

The V − A character of the τ–ν_τ and e–ν_e currents is shown in Fig. 13.4a. In the τ rest frame, when the electron energy is maximum ($E \approx E_{\max}$), kinematics implies that the two neutrinos ν_τ and $\overline{\nu}_e$ are emitted in one direction (for instance the $+x$ axis), whereas the electron is emitted in the opposite direction ($-x$ axis). Since the two neutrinos have opposite helicities, the total angular momentum conservation in the x axis forces the spin of e^- to be parallel to the spin of τ^-. Since the electron has negative helicity at high momentum, it must be emitted antiparallel to the τ^- spin.

This correlation between spin and three-momentum is also described by the asymmetry parameter α in (20). When $E = E_{\max} = M/2$, $\alpha = +1$, and the electron is likely emitted antiparallel to the τ^- spin direction, whereas the positron prefers to be emitted parallel to the τ^+ spin. This $E_{\max}$ configuration is illustrated in Fig. 13.4.

Near the lower end $E = 0$, exactly the opposite configuration appears, since $\alpha = -\frac{1}{3}$.

Moreover, Fig. 13.4 shows why the spin and momentum correlation in the τ^+ decay can be derived from that of the τ^-, assuming CP invariance of the weak interaction involving leptons. Starting with the $\tau^- \to \nu_\tau + e^- + \overline{\nu}_e$ decay in Fig. 13.4a, let us consider its charge conjugate states in Fig. 13.4b. The configuration of the latter cannot be observed because all of the e^+, ν_e and $\overline{\nu}_\tau$ have the wrong helicities. The charge conjugation symmetry C is manifestly violated by weak interactions. We then go further by letting the space reversal $\mathcal{P}$ operate on Fig. 13.4b, which becomes Fig. 13.4c. The combination of $\mathcal{C}$ and $\mathcal{P}$, i.e. $\mathcal{CP}$, transforms Fig. 13.4a into Fig. 13.4c. While the violations of P and C are strongest possible, the product CP is to a good approximation conserved by weak interactions. If CP is invariant, Fig. 13.4c must be physically observable. The angular distribution of the e^+, with respect to the τ^+ polarization, can be obtained from that of the e^- in τ^- decay, by a simple change of sign $\cos\theta \leftrightarrow -\cos\theta$ in (24). This substitution rule $\cos\theta \leftrightarrow -\cos\theta$ in $\tau^- \leftrightarrow \tau^+$ may be used as a test of CP violation.

Finally, the energy spectrum and the width are given by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E} = \frac{G_{\mathrm{F}}^2\,|\boldsymbol{p}|}{12\,\pi^3}\Big[ME(3M-4E) - m^2(2M-3E)\Big]\,, \tag{13.25}$$

$$\Gamma = \frac{G_{\mathrm{F}}^2 M^5}{192\,\pi^3} f\left(\frac{m^2}{M^2}\right) = \Gamma_0\, f\left(\frac{m^2}{M^2}\right)\ . \tag{13.26}$$

The integrated width Γ depends on *the fifth power of the energy released by the decaying particle* (which is M, in the case of massless decay products). This power law is easy to understand since the dimension of G_{F}^2 is $(\mathrm{mass})^{-4}$ and that of the width is $(\mathrm{mass})^{+1}$. For a fermion of mass M decaying into three massless fermions, the only mass involved is M, so $G_{\mathrm{F}}^2 M^5$ naturally appears. The huge difference by a factor of 10^{16} in the lifetimes of weakly decaying particles (for instance between the charm D meson and the neutron) essentially comes from this fifth power of the energy released. For neutron, the energy liberated $\approx m_{\mathrm{n}} - m_{\mathrm{p}} - m_{\mathrm{e}}$ is only 0.78 MeV, whereas for charmed or bottom-flavored mesons, it can reach a few GeVs. A more accurate estimate calls for more sophisticated computations, however this fifth power can explain the huge differences in the lifetimes of weakly decaying particles.

Beside these tree diagram results, we should add the electromagnetic radiative corrections to leptonic weak interactions. These corrections are due to virtual photons in loops involving the charged τ and ℓ, as well as to real photons emitted by them (bremsstrahlung). There are in all five diagrams similar to the five drawn in Figs. 14.2–3 with photons replacing gluons. The calculation could be done similarly to that in Chap. 14. These corrections[3] yield

$$1 - \frac{\alpha_{\mathrm{em}}}{2\pi}\left(\pi^2 - \frac{25}{4}\right)\ . \tag{13.27}$$

Another type of corrections involves the W propagator effect if we do not neglect $q^2 \ll M_{\mathrm{W}}^2$ in (1). This gives $1 + \frac{3}{5}\frac{M^2}{M_{\mathrm{W}}^2} - 2\frac{m^2}{M_{\mathrm{W}}^2}$. These two types of corrections multiplied by (21) give

$$\Gamma = a^2\Gamma_0\, f\left(\frac{m^2}{M^2}\right)\left[1 - \frac{\alpha_{\mathrm{em}}}{2\pi}\left(\pi^2 - \frac{25}{4}\right)\right]\left[1 + \frac{3}{5}\frac{M^2}{M_{\mathrm{W}}^2} - 2\,\frac{m^2}{M_{\mathrm{W}}^2}\right]\ . \tag{13.28}$$

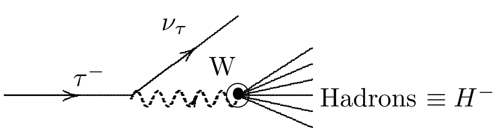
[3] Berman, S., Phys. Rev. **112** (1958) 267; Kinoshita, T. and Sirlin, A., Phys. Rev. **113** (1959) 1652

These last two corrections are numerically small $\approx 4 \times 10^{-3}$, i.e. the product of the last two factors of the above equation is 0.996. We note, however, that one-loop QCD radiative corrections – relevant to the inclusive semileptonic τ decay described by $\tau \to \nu_\tau +$ a quark pair $(\mathrm{q_i} + \overline{\mathrm{q}}_j)$, in which gluons replace photons – are much more important, simply because $(-\alpha_{\rm em}/2\pi) \times (\pi^2 - 25/4)$ is replaced by $+\alpha_s/\pi$, where the running coupling $\alpha_s(M)$ turns out to be ≈ 0.37 at the appropriate scale M of the τ decays. As we will see in Chap. 14, these one-loop QCD corrections, which enhance the $\Gamma(\tau \to \nu_\tau + \mathrm{q}_i + \overline{\mathrm{q}}_j)$ rate, will consequently pull down the leptonic branching ratio from its naive 0.2 value (Sect. 13.2) to 0.186, closer to the observed $\mathrm{B_e} = 0.1783 \pm 0.0006$.

Finally, we note that (24)–(26) and (28) also apply to $\mu^- \to \nu_\mu + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$, for which M and m are the muon and electron masses respectively.

13.3 Semileptonic Decays

The τ is the only lepton massive enough to decay into hadrons. Its semileptonic channels in both exclusive and inclusive modes are ideal for studying strong interaction in clean conditions.



$$\langle H^- \,|\, \mathcal{V}_\mu \,|\, 0\rangle \;\; ; \qquad \langle H^- \,|\, \mathcal{A}_\mu \,|\, 0\rangle$$

$$\mathcal{V}_\mu = V_{\rm ud}\, \overline{d}\gamma_\mu u + V_{\rm us}\, \overline{s}\gamma_\mu u$$

$$\mathcal{A}_\mu = V_{\rm ud}\, \overline{d}\gamma_\mu\gamma_5 u + V_{\rm us}\, \overline{s}\gamma_\mu\gamma_5 u$$

Fig. 13.5. Semileptonic decays of τ^-

These decays probe the matrix element of the V and A parts of the charged current between the vacuum and the final hadronic state H (Fig. 13.5). Since these matrix elements are just the decay constants (if H is a single particle) or form factors (if H represents several particles), the importance of semileptonic decays cannot be underrated. The two-pion modes [including the $\rho(770)$] and more gererally the $2n$-pion modes are the cleanest hadronic channels in which the CVC property of the charged current $\overline{d}\gamma^\mu u$ (a consequence of its isospin structure) can be unambiguously tested at relatively high momentum transfer q^2 released by τ.

13.3.1 The One-Pion Mode: $\tau^- \to \nu_\tau + \pi^-$

The relevant hadronic current sandwiched between a pseudoscalar pion and the vacuum is $V_{\rm ud}\, A_\mu = V_{\rm ud}\, \overline{d}\gamma_\mu\gamma_5 u$ and $\langle \pi^-(p) \,|\, A_\mu \,|\, 0\rangle \equiv \mathrm{i}\, f_\pi p_\mu$. We note that the pion decay constant f_π is the same parameter that enters the π_{ℓ_2} mode $\pi \to \ell + \overline{\nu}_\ell$ of Fig. 10.3a, and f_π is one of the most fundamental constants frequently met in different circumstances in particle physics.

The amplitude $\tau^-(P) \to \nu_\tau(k) + \pi^-(p)$ can be written as

$$\begin{aligned}\mathcal{M} =& \Big(\frac{-\mathrm{i}g}{2\sqrt{2}}\Big)^2 \overline{u}(k)\gamma_\nu(1-\gamma_5)u(P)\frac{-\mathrm{i}(g^{\mu\nu} - \frac{p^\mu p^\nu}{M_W^2})}{p^2 - M_W^2}\ V_{\mathrm{ud}}\ \langle \pi^-(p)\,|\,A_\mu\,|\,0\rangle \\ =& \frac{G_{\mathrm{F}}}{\sqrt{2}} V_{\mathrm{ud}}\, M\, f_\pi\, \overline{u}(k)(1+\gamma_5)u(P)\ . \end{aligned} \tag{13.29}$$

Averaging the intial state τ spin, we get

$$\tfrac{1}{2}\sum_{\mathrm{spin}} |\mathcal{M}|^2 = \frac{G_{\mathrm{F}}^2}{2} |V_{\mathrm{ud}}|^2\, M^2\, f_\pi^2\, \mathrm{Tr}\Big[\not{k}\,\not{P}(1-\gamma_5)\Big] = G_{\mathrm{F}}^2 |V_{\mathrm{ud}}|^2 f_\pi^2 M^4 \left(1 - \frac{m_\pi^2}{M^2}\right).$$

Applying (4.73) for the decay with two particles in the final state, we have

$$\Gamma(\tau \to \nu_\tau + \pi) = \frac{1}{2\,M} \int \frac{\mathrm{d}^3 k}{2E_k}\frac{\mathrm{d}^3 p}{2E_p}\frac{\delta^4(k+p-P)}{(2\pi)^2}\ \tfrac{1}{2}\sum_{\mathrm{spin}} |\mathcal{M}|^2\ .$$

Using the two-body phase space integral formula in the Appendix,

$$\int \frac{\mathrm{d}^3 k}{2E_k}\frac{\mathrm{d}^3 p}{2E_p}\delta^4(k+p-P) = \frac{\pi}{2}\frac{\sqrt{\lambda(P^2, 0, p^2)}}{P^2} = \frac{\pi}{2}\left(1 - \frac{m_\pi^2}{M^2}\right)\ , \tag{13.30}$$

we obtain

$$\Gamma = \frac{G_{\mathrm{F}}^2 |V_{\mathrm{ud}}|^2}{16\pi}\, f_\pi^2\, M^3 \Big(1 - \frac{m_\pi^2}{M^2}\Big)^2 = 12\,\pi^2\, |V_{\mathrm{ud}}|^2\, \frac{f_\pi^2}{M^2}\Big(1 - \frac{m_\pi^2}{M^2}\Big)^2\, \Gamma_0\ . \tag{13.31}$$

This formula is to be compared with $\pi^- \to \mu^- + \overline{\nu}_\mu$ (Problem 5.3):

$$\Gamma(\pi^- \to \ell^- + \overline{\nu}_\ell) = \frac{G_{\mathrm{F}}^2\, |V_{\mathrm{ud}}|^2}{8\pi} f_\pi^2 m_\ell^2\, m_\pi \Big(1 - \frac{m_\ell^2}{m_\pi^2}\Big)^2\ . \tag{13.32}$$

From $\Gamma(\pi^- \to \mu^- + \overline{\nu}_\mu) = 0.384 \times 10^8\ \mathrm{s}^{-1} = 2.53 \times 10^{-14}$ MeV, one gets $f_\pi \approx 131$ MeV. Comparing (26) with (31), the ratio of the branching fractions

$$\frac{\mathrm{B}_\pi}{\mathrm{B}_{\mathrm{e}}} \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \pi^-)}{\Gamma(\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_{\mathrm{e}})} = 12\pi^2\, |V_{\mathrm{ud}}|^2 \frac{f_\pi^2}{M^2}\Big(1 - \frac{m_\pi^2}{M^2}\Big)^2 = 0.60$$

is in good agreement with data. A straightforward generalization can be made for the Cabibbo-suppressed mode $\tau^- \to \nu_\tau + \mathrm{K}^-$, with the interchange $V_{\mathrm{ud}} \leftrightarrow V_{\mathrm{us}}$ and $f_\pi, m_\pi \leftrightarrow f_{\mathrm{K}}, m_{\mathrm{K}}$ in (31). The decay constant $f_{\mathrm{K}} \approx 160$ MeV is obtained from the $\mathrm{K}^- \to \mu^- + \overline{\nu}_\mu$ rate, similar to (32) for f_π. The ratio

$$\frac{\mathrm{B}_{\mathrm{K}}}{\mathrm{B}_\pi} \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \mathrm{K}^-)}{\Gamma(\tau^- \to \nu_\tau + \pi^-)} = \frac{|V_{\mathrm{us}}|^2}{|V_{\mathrm{ud}}|^2}\frac{f_{\mathrm{K}}^2}{f_\pi^2}\Big(\frac{M^2 - m_{\mathrm{K}}^2}{M^2 - m_\pi^2}\Big)^2 = 0.066$$

is in agreement with data.

13.3.2 The $2n$-Pion Mode and CVC

Before the discovery of τ, CVC was only tested at low momentum transfer q^2 in nuclear physics and neutrino–nucleon scattering (Chap. 12). For the first time CVC can be tested at high momentum q^2 released by τ. The $\tau^-(P) \to \nu_\tau(k) + \pi^-(p_1) + \pi^0(p_2)$ amplitude can be obtained from the weak vector current $V_\mu = \overline{d}\gamma_\mu u$:

$$\mathcal{M} = \frac{G_{\rm F}\, V_{\rm ud}}{\sqrt{2}} \overline{u}(k)\gamma^\mu(1-\gamma_5)u(P)\ \big\langle \pi^-(p_1)\pi^0(p_2)\,|\, V_\mu \,|\, 0\big\rangle\ . \tag{13.33}$$

Using CVC, we can relate the two-pion matrix element of V_μ to that of the electromagnetic current $J_\mu^{\rm em}$:

$$\begin{aligned}\big\langle \pi^-(p_1)\pi^0(p_2)\,|\, V_\mu \,|\, 0\big\rangle &= \sqrt{2}\ \big\langle \pi^-(p_1)\pi^+(p_2)\,\big|\, J_\mu^{\rm em} \,\big|\, 0\big\rangle \\ &= \sqrt{2}\,(p_1-p_2)_\mu F_\pi(q^2)\ ,\end{aligned} \tag{13.34}$$

where $q_\mu = (p_1+p_2)_\mu$. $F_\pi(q^2)$ is the pion electromagnetic form factor, already introduced in (10.11). The momentum transfer q^2 is $\geq 4m_\pi^2$. We calculate

$$\begin{aligned}&\tfrac{1}{2}\sum_{\rm spin} |\mathcal{M}|^2 = 4\,G_{\rm F}^2|V_{\rm ud}|^2|F_\pi(q^2)|^2\,\big\{Y\big\}\ ,\\ &\big\{Y\big\} = 2\,[k\cdot(p_1-p_2)]\,[P\cdot(p_1-p_2)] - (P\cdot k)(p_1-p_2)^2\ .\end{aligned} \tag{13.35}$$

Using $P = k+q$ and $P\cdot(p_1-p_2) = k\cdot(p_1-p_2)$, we rewrite $\big\{Y\big\}$ in the following form which is convenient for the phase space integration:

$$\big\{Y\big\} = 6\,(k\cdot p_1)^2 + 2\,(k\cdot p_2)^2 - 2\,(M^2-q^2)\,(k\cdot p_1) + \frac{(M^2-q^2)\,(q^2-4m_\pi^2)}{2}\ .$$

With (35), the decay rate is given by

$$\begin{aligned}&\Gamma = 4\,G_{\rm F}^2|V_{\rm ud}|^2\frac{1}{2M}\ \frac{1}{(2\pi)^5}\int_{\mathcal{PS}3}\big\{Y\big\}|F_\pi(q^2)|^2\ ,\\ &\text{where}\ \int_{\mathcal{PS}3} \equiv \int \frac{{\rm d}^3p_1}{2E_1}\frac{{\rm d}^3p_2}{2E_2}\frac{{\rm d}^3k}{2E}\delta^4(p_1+p_2+k-P)\ .\end{aligned} \tag{13.36}$$

Since (35) is symmetric under $p_1 \leftrightarrow p_2$, the phase space integration $\int_{\mathcal{PS}3}$ of $(k\cdot p_2)^2$ in $\big\{Y\big\}$ is equal to that of $(k\cdot p_1)^2$. With the help of formulas in the Appendix, we get for different terms in $\big\{Y\big\}$:

$$\begin{aligned}&\int_{\mathcal{PS}3}(k\cdot p_1)^2 = \frac{\pi^2}{48}\int_{4m_\pi^2}^{M^2}{\rm d}q^2\left(1-\frac{q^2}{M^2}\right)\sqrt{1-\frac{4m_\pi^2}{q^2}}\,(1-\frac{m_\pi^2}{q^2})\,\big(M^2-q^2\big)^2\,,\\ &\int_{\mathcal{PS}3}(k\cdot p_1) = \frac{\pi^2}{16}\int_{4m_\pi^2}^{M^2}{\rm d}q^2\ \left(1-\frac{q^2}{M^2}\right)\sqrt{1-\frac{4m_\pi^2}{q^2}}(M^2-q^2)\ ,\\ &\int_{\mathcal{PS}3} = \frac{\pi^2}{4}\int_{4m_\pi^2}^{M^2}{\rm d}q^2\left(1-\frac{q^2}{M^2}\right)\sqrt{1-\frac{4m_\pi^2}{q^2}}\ .\end{aligned} \tag{13.37}$$

Putting together (35) and (37) into (36), we finally obtain

$$\Gamma = \frac{G_F^2\,|V_{ud}|^2\,M^3}{384\,\pi^3}\int_{4m_\pi^2}^{M^2} \mathrm{d}\,q^2 \left(1-\frac{4m_\pi^2}{q^2}\right)^{3/2}\left(1-\frac{q^2}{M^2}\right)^2\left(1+\frac{2\,q^2}{M^2}\right)|F_\pi(q^2)|^2$$
$$= \frac{\Gamma_0\,|V_{ud}|^2}{2}\int_{4m_\pi^2}^{M^2} \frac{\mathrm{d}\,q^2}{M^2}\left(1-\frac{4m_\pi^2}{q^2}\right)^{3/2}\left(1-\frac{q^2}{M^2}\right)^2\left(1+\frac{2\,q^2}{M^2}\right)|F_\pi(q^2)|^2\,.$$

The pion form factor $|F_\pi(q^2)|$ can be directly measured from experiments $e^+ + e^- \to \pi^+ + \pi^-$, its cross-section is given by (Problem 10.3)

$$\sigma_{e^+ + e^- \to \pi^+ + \pi^-}(q^2) = \frac{\pi\,\alpha^2}{3\,q^2}\left(1-\frac{4m_\pi^2}{q^2}\right)^{3/2}|F_\pi(q^2)|^2 \equiv \sigma(q^2)\,, \tag{13.38}$$

then $$\Gamma = \Gamma_0\,\frac{3\,|V_{ud}|^2}{2\pi\,\alpha^2}\int_{4m_\pi^2}^{M^2}\frac{\mathrm{d}\,q^2}{M^2}\left(1-\frac{q^2}{M^2}\right)^2\left(1+\frac{2\,q^2}{M^2}\right)q^2\,\sigma(q^2)\,. \tag{13.39}$$

Using the data[4] for $\sigma(q^2)$ in (39) and performing numerical integration, the resulting branching ratio $B_{2\,\pi} = (23.54 \pm 1.2)\%$ is in agreement with experiment $(25.24 \pm 0.16)\%$. Using only the $e^+ + e^- \to \pi^+ + \pi^-$ data as input, this result constitutes a powerful test of CVC.

Also, for an even number of pions, the rate $\Gamma(\tau \to \nu_\tau + 2n$ pions) can be directly obtained by CVC from the cross-sections $\sigma(e^+ + e^- \to 2n$ pions) using (39). In particular the branching ratios for $\tau^- \to \nu_\tau + \pi^- + \pi^+ + \pi^- + \pi^0$ and $\tau^- \to \nu_\tau + \pi^- + \pi^0 + \pi^0 + \pi^0$ are computed to be respectively $(4.9 \pm 2)\%$ and $(0.98 \pm 0.4)\%$, using $\sigma(e^+ + e^- \to 4$ pions) data. They are again in good agreement with experiments.

We note that the two pions must be in an isospin $I = 1$ state since it is created from the vacuum by the $I = 1$ vector current $\overline{d}\gamma^\mu u$. Bose statistics implies that the dipion is in p-wave. It turns out that in the energy range ~ 1 GeV of τ decay, the p-wave two-pion state resonates to form the $\rho(770)$ meson: $\tau^- \to \nu_\tau + \pi^- + \pi^0 \approx \tau^- \to \nu_\tau + \rho^-$. This $\rho(770)$ dominance of the two-pion state enhances the pion form factor $F_\pi(q^2)$ in the region $q^2 \approx m_\rho^2$. As discussed in Chap. 10 (Fig. 10.2), the form factor $F_\pi(q^2)$ may be parameterized by the Breit–Wigner resonance form

$$F_\pi(q^2) = \frac{m_\rho f_\rho g_{\rho\pi\pi}}{m_\rho^2 - q^2 - \mathrm{i}\,m_\rho\Gamma_\rho} \longrightarrow \frac{m_\rho f_\rho g_{\rho\pi\pi}}{m_\rho^2 - q^2 - \mathrm{i}\sqrt{q^2}\,\Gamma_\rho(q^2)}\,. \tag{13.40}$$

In (40), instead of keeping the constant $\mathrm{i}\,m_\rho\Gamma_\rho$, it may be more appropriate to take into account the q^2 dependence of the ρ width, i.e.

$$m_\rho\Gamma_\rho \longrightarrow \sqrt{q^2}\,\Gamma_\rho(q^2)\,,\ \text{where}\ \Gamma_\rho(q^2) = \Gamma_\rho\frac{m_\rho^2}{q^2}\left(\frac{q^2-4m_\pi^2}{m_\rho^2-4m_\pi^2}\right)^{3/2}. \tag{13.41}$$

[4] L.M. Barkov et al., Nucl. Phys. **B256** (1985) 365

Putting (40) into (38), after doing the q^2 integration, the rate obtained is again in excellent agreement with data. In the narrow width approximation of the ρ, the Breit–Wigner factor becomes a delta function. Using

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \frac{1}{x^2 + \varepsilon^2} \ ,$$

we write

$$|F_\pi(q^2)|^2 = \frac{\left(m_\rho\, f_\rho g_{\rho\pi\pi}\right)^2}{(q^2 - m_\rho^2)^2 + \Gamma_\rho^2\, m_\rho^2} \longrightarrow \frac{\pi\ \delta(q^2 - m_\rho^2)}{\Gamma_\rho\, m_\rho} \left(m_\rho\, f_\rho g_{\rho\pi\pi}\right)^2 . \quad (13.42)$$

We recall that the strong coupling $g_{\rho\pi\pi}$ is related to the ρ width Γ_ρ by (10.20)

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2\, m_\rho}{48\pi} \Big(1 - \frac{4m_\pi^2}{m_\rho^2}\Big)^{3/2} \ .$$

Combining the above relation with (38) and (42), we get

$$\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \rho^0) \longrightarrow 16\alpha^2\pi^3 \frac{f_\rho^2}{m_\rho^2} \delta(q^2 - m_\rho^2) \ . \quad (13.43)$$

Putting (43) in (39), we obtain

$$\Gamma(\tau^- \to \nu_\tau + \rho^-) = 24\,\pi^2\, |V_{\mathrm{ud}}|^2 \frac{f_\rho^2}{M^2} \left(1 + \frac{2m_\rho^2}{M^2}\right) \left(1 - \frac{m_\rho^2}{M^2}\right)^2 \Gamma_0 \ . \quad (13.44)$$

According to (10.21) and Fig. 10.3, the ρ^0 decay constant is $f_\rho \approx 150 \pm 10$ MeV coming from the $\rho^0 \to \mathrm{e}^+ + \mathrm{e}^-$ width. We get

$$\begin{aligned} \frac{\mathrm{B}_\rho}{\mathrm{B}_\mathrm{e}} &\equiv \frac{\Gamma(\tau^- \to \nu_\tau + \rho^-)}{\Gamma(\tau^- \to \nu_\tau + \mathrm{e}^- + \overline{\nu}_\mathrm{e})} = 24\pi^2 |V_{\mathrm{ud}}|^2 \frac{f_\rho^2}{M^2} \Big(1 + \frac{2m_\rho^2}{M^2}\Big)\Big(1 - \frac{m_\rho^2}{M^2}\Big)^2 \\ &= 1.44 \pm 0.2 \qquad \Longrightarrow \mathrm{B}_\rho = 0.256 \pm 0.035 \ . \end{aligned} \quad (13.45)$$

A straightforward generalization can also be made for the Cabibbo-suppressed mode $\tau^- \to \nu_\tau + \mathrm{K}^{*-}(892)$ by the substitution $V_{\mathrm{ud}} \to V_{\mathrm{us}}\,,\ m_\rho \to m_{\mathrm{K}^*}$, $f_\rho \to f_{\mathrm{K}^*}$ in (44). However, the decay constant f_{K^*}, unlike the f_ρ, is difficult to determine by experiment. It may be estimated by the SU(3) flavor symmetry, which gives $m_{\mathrm{K}^*}\, f_{\mathrm{K}^*} = m_\rho\, f_\rho$. We get $\mathrm{B}_{\mathrm{K}^*} = (1.1 \pm 0.1)\%$ for the branching ratio of $\tau \to \nu_\tau + \mathrm{K}^*$ to be compared with data $(1.43 \pm 0.31)\%$.

The method for obtaining $\tau^- \to \nu_\tau + \pi^- + \pi^0$ can be generalized to the $\tau^- \to \nu_\tau + \mathrm{K}^- + \mathrm{K}^0$ decay, which is also a Cabibbo-favored mode. The $\mathrm{K}^- + \mathrm{K}^0$ pair $(S = 0)$ can be created by the same conserved vector current $\overline{d}\gamma^\mu u$ from the sea $\overline{\mathrm{s}}\mathrm{s}$ in the vacuum . The only replacements in (38) and (39) are $F_\pi(q^2) \leftrightarrow F_\mathrm{K}(q^2)$ and $m_\pi \leftrightarrow m_\mathrm{K}$. However, compared with the two-pion case, the $\mathrm{K}^- + \mathrm{K}^0$ mode is suppressed by both kinematic and dynamic reasons. Kinematic suppression is due to the factor $\frac{1}{2}\,[1 - (4m_\mathrm{K}^2/q^2)]^{3/2}$, and the q^2 phase space integration range is smaller. Dynamic suppression occurs because an $I = 1, J^P = 1^-$ resonance for $\mathrm{K} + \overline{\mathrm{K}}$ is lacking, the form factor $F_\mathrm{K}(q^2)$ is not enhanced and is smaller than $F_\pi(q^2)$. The sea $\overline{\mathrm{s}}\mathrm{s}$ contribution is also negligible.

13.4 The Method of Spectral Functions

Instead of considering semileptonic exclusive channels with one or two particles in the final state as we have just done, we introduce now the notion of spectral functions which represent a more systematic way of dealing with multiparticle system. The aim is to derive a general formulation, valid for any hadronic channels, and all the formulas developed in the previous section could be recovered. In addition, the formalism is useful later for the computation of the inclusive rate.

The semileptonic decay width $\tau^-(P) \to \nu_\tau(k) + H^-$, where H^- is any hadronic system of one or several particles, can be written as

$$\Gamma(\tau^- \to \nu_\tau + H^-) = \frac{1}{2M}\frac{G_{\rm F}^2}{2}\int \frac{{\rm d}^3 k}{2E_k(2\pi)^3}\,\frac{1}{2}\big\{2\,{\rm Tr}[\not{k}\gamma_\mu \not{P}\gamma_\nu(1-\gamma_5)]\big\}\mathcal{H}^{\mu\nu},$$
$$\mathcal{H}^{\mu\nu} \equiv \sum_{PS_{\rm H}} \big\langle 0\,|\,J^\mu\,|\,H^-(p_{\rm H})\big\rangle \big\langle H^-(p_{\rm H})\,\big|\,J^{\dagger\nu}\,\big|\,0\big\rangle\,(2\pi)^4\delta^4(p_{\rm H}+k-P). \quad (13.46)$$

In (46), the coefficient $\frac{1}{2}$ represents the averaging of the τ spin, while the coefficient 2 before the trace comes from $(1-\gamma_5)^2 = 2(1-\gamma_5)$ of the leptonic tensor. Similar to (10.42), the hadronic tensor $\mathcal{H}^{\mu\nu}$ is the product of the matrix element of the current J^μ with that of the current $J^{\nu\dagger}$, where the V − A current J^μ is the sum of two parts $V_{\rm ud}\,\overline{d}\gamma^\mu(1-\gamma_5)u$ and $V_{\rm us}\,\overline{s}\gamma^\mu(1-\gamma_5)u$. The first term $V_{\rm ud}\,\overline{d}\gamma^\mu(1-\gamma_5)u$ carries isospin $I=1$ and strangeness $S=0$, while the second term $V_{\rm us}\overline{s}\gamma^\mu(1-\gamma_5)u$ has $I=1/2$ and $S=-1$. The symbol $\sum_{PS_{\rm H}}$ of $\mathcal{H}^{\mu\nu}$ in (46) denotes the phase space integration of the final state in H^- (including the spin summation):

$$\sum_{PS_{\rm H}} \equiv \int \prod_{j\in H}\Big[\frac{{\rm d}^3 p_j}{2E_j(2\pi)^3}\Big]\ ,\ \text{with}\ \ p_{\rm H} \equiv \sum_j p_j = P - k \equiv q\ .$$

No matter how complicated $\mathcal{H}^{\mu\nu}$ is, it has two general features:

(i) $\mathcal{H}^{\mu\nu}$ has the dimension of (mass)^2, we can check this point by inspecting the dimension of the different terms in (46).

(ii) After doing the phase space integration of all the particles in H^- and summing over their spins, the four-momentum transfer q remains as the only dynamical variable in $\mathcal{H}^{\mu\nu}(q)$. Therefore the latter must be a function of the Lorentz-invariant q^2 and can only depend on the tensors $q^\mu q^\nu$ and $g^{\mu\nu}$. They can be conveniently put into two independent sets $(-q^2 g^{\mu\nu} + q^\mu q^\nu)$ and $q^\mu q^\nu$ which are respectively orthogonal and parallel to the four-momentum q, i.e. the former satisfies $q_\mu(-q^2 g^{\mu\nu} + q^\mu q^\nu) = 0$, $q_\nu(-q^2 g^{\mu\nu} + q^\mu q^\nu) = 0$. It is important to note that the hadronic state H^- (whether a single or multiparticle system) has a total angular momentum $j=1$ or 0, since H^- is created from the vacuum by the virtual W boson which carries at most $j=1$.

These two features enable us to write the most general form for $\mathcal{H}^{\mu\nu}$ in terms of the dimensionless spectral functions

$$\begin{aligned}\sum_{PS_{\mathrm{H}}} &\left\langle 0 \,|\, J^\mu \,|\, H^-(p_{\mathrm{H}}) \right\rangle \left\langle H^-(p_{\mathrm{H}}) \,\middle|\, J^{\nu\dagger} \,\middle|\, 0 \right\rangle (2\pi)^4 \delta^4(p_{\mathrm{H}} - q) \\ &= (-q^2 g^{\mu\nu} + q^\mu q^\nu) \left[v_1(q^2) + a_1(q^2)\right] + q^\mu q^\nu \left[v_0(q^2) + a_0(q^2)\right] .\end{aligned} \tag{13.47}$$

The $v_{1,0}(q^2)$ come from the vector currents, the $a_{1,0}(q^2)$ from the axial currents. The subscript 1 in $v_1(q^2)$ and $a_1(q^2)$ corresponds to the total angular momentum $j = 1$ of H^-, whereas $v_0(q^2)$ and $a_0(q^2)$ are associated with $j = 0$. Note that the vector current $V^\mu = \overline{d}\gamma^\mu u$ is conserved ($q_\mu \overline{d}\gamma^\mu u = 0$), the product of the matrix element of V^μ with that of $V^{\nu\dagger}$ gives rise, after the phase space $\sum_{PS_H}$ operation, to the tensor $(-q^2 g^{\mu\nu} + q^\mu q^\nu)$. This implies that only $v_1(q^2)$ exists, while $v_0(q^2) = 0$.

On the other hand, the axial current $A^\mu = \overline{d}\gamma^\mu\gamma_5 u$ is not conserved, the product of the matrix element of A^μ with that of $A^{\nu\dagger}$ contains both $(-q^2 g^{\mu\nu} + q^\mu q^\nu) a_1(q^2)$ and $q^\mu q^\nu a_0(q^2)$. Because of the G-parity, the current V^μ (A^μ) produces an even (odd) numbers of pions in the final state. Table 13.1 summarizes the point.

Table 13.1. Spectral functions of currents and the associated final states

Currents	Spectral functions	J^P	Final states
$V^\mu = \overline{d}\gamma^\mu u$	$v_1(q^2)$, $v_0(q^2) = 0$	1^-	ρ^- , 2π , 4π , $\mathrm{K}^- + \mathrm{K}^0$
$A^\mu = \overline{d}\gamma^\mu\gamma_5 u$	$a_1(q^2)$, $a_0(q^2)$	0^- , 1^+	π^- , 3π , $\mathrm{a}_1^-(1260)$
$V_{\mathrm{S}}^\mu = \overline{s}\gamma^\mu u$	$v_1^{\mathrm{S}}(q^2)$, $v_0^{\mathrm{S}}(q^2)$	0^+ , 1^-	$\mathrm{K}_0^{*-}(1430)$, $\mathrm{K}^{*-}(892)$
$A_{\mathrm{S}}^\mu = \overline{s}\gamma^\mu\gamma_5 u$	$a_1^{\mathrm{S}}(q^2)$, $a_0^{\mathrm{S}}(q^2)$	0^- , 1^+	K^- , $\mathrm{K}_1^-(1270)$

Since (47) is symmetric in $\mu \leftrightarrow \nu$, when we put $\mathcal{H}^{\mu\nu}$ into (46), the role of γ_5 in the leptonic part $\mathrm{Tr}[\not{k}\gamma_\mu \not{P}\gamma_\nu(1-\gamma_5)]$ is automatically superfluous, its antisymmetric tensor $\mathrm{i}\,\varepsilon_{\mu\nu\alpha\beta}\, k^\alpha P^\beta$ does not contribute. Note that

$$\int \frac{\mathrm{d}^3 k}{2E_k} = \frac{\pi}{2M^2} \int \sqrt{\lambda(M^2, k^2, q^2)}\, \mathrm{d}q^2 \underset{k^2=0}{\longrightarrow} \frac{\pi}{2M^2} \int (M^2 - q^2)\, \mathrm{d}q^2 \; . \tag{13.48}$$

With massless neutrino ($k^2 = 0$), using (46), (47), and (48), we obtain for the Cabibbo-favored modes associated with V_{ud} and the current $\overline{d}\gamma^\mu(1-\gamma_5)u$

$$\begin{aligned}\Gamma^{S=0} =& \frac{G_{\mathrm{F}}^2 |V_{\mathrm{ud}}|^2}{32\pi^2 M^3} \int_{m_\pi^2}^{M^2} \mathrm{d}q^2\, (M^2 - q^2)^2 \\ &\times \Big\{ (M^2 + 2q^2)[v_1(q^2) + a_1(q^2)] + M^2 a_0(q^2) \Big\}.\end{aligned} \tag{13.49}$$

Associated with $V_{\rm us}$, the vector part $\overline{s}\gamma^\mu u$ of the $I=\frac{1}{2}, S=-1$ current $\overline{s}\gamma^\mu(1-\gamma_5)u$ is not conserved, so the corresponding spectral function $v_0^S(q^2)$ is not vanishing. This Cabibbo-suppressed width is

$$\Gamma^{S=-1} = \frac{G_{\rm F}^2|V_{\rm us}|^2}{32\pi^2\, M^3}\int_{m_{\rm K}^2}^{M^2} {\rm d}q^2\,(M^2-q^2)^2 \times \Big\{(M^2+2q^2)[v_1^S(q^2)+a_1^S(q^2)] + M^2[v_0^S(q^2)+a_0^S(q^2)]\Big\}\,. \tag{13.50}$$

Formulas (49) and (50) show that the calculation of the rate is reduced to a computation of the spectral functions.

We can check the general structure (47) of $\mathcal{H}^{\mu\nu}(q)$ with one-particle state by putting $H^-=\pi^-\,,\,\rho^-$ in (47) and find

$$a_0^\pi(q^2) = 2\pi f_\pi^2\,\delta(q^2-m_\pi^2) \quad ; \quad v_1^\rho(q^2) = 2\pi(\sqrt{2}f_\rho)^2\,\delta(q^2-m_\rho^2)\,. \tag{13.51}$$

The easiest way to obtain $a_0^\pi(q^2)$ in (51) is to use

$$\langle 0\,|\,A^\mu\,|\,\pi(p_{\rm H})\rangle = {\rm i}\,f_\pi\,p_{\rm H}^\mu \;\;\text{and}\;\; \frac{{\rm d}^3p_{\rm H}}{(2\pi)^3\,2E_{\rm H}} = \frac{{\rm d}^4p_{\rm H}}{(2\pi)^3}\,\delta(p_{\rm H}^2-m_\pi^2)\theta(p_{\rm H}^0)\,.$$

The tensor $(-q^2g^{\mu\nu}+q^\mu q^\nu)$ associated with $v_1^\rho(q^2)$ comes from

$$\sum_{\rho\ \rm spin}\varepsilon^\mu(q)\,\varepsilon^\nu(q) = -g^{\mu\nu}+q^\mu\,q^\nu/m_\rho^2\,.$$

Putting (51) into (49), we recover (31) and (44).

For the two-pion state $\pi^-+\pi^0$, the corresponding spectral function $v_1^{\pi\pi}(q^2)$ can be obtained by putting (34) into (47) and using the formulas in the Appendix for the two-particle phase space integration. We get

$$v_1^{\pi\pi}(q^2) = \Big(1-\frac{4m_\pi^2}{q^2}\Big)^{3/2}\frac{|F_\pi(q^2)|^2}{12\pi}\,. \tag{13.52}$$

Putting (52) into (49), we again recover (38) and (39).

13.4.1 The Three-Pion Mode

As a first illustration of the spectral function method, let us consider the $\tau^-\to\nu_\tau+3\pi$ decay which has a substantial branching ratio of about 18% for the sum of two modes $\pi^-+\pi^++\pi^-$ and $\pi^-+\pi^0+\pi^0$. By the G-parity conservation, the vector current $V^\mu=\overline{d}\gamma^\mu u$, which has G=+1, does not contribute. The relevant axial current $A^\mu=\overline{d}\gamma^\mu\gamma_5 u$, responsible for $\tau\to\nu_\tau+3\,\pi$, is the same current that intervenes in the one-pion mode discussed previously. However, in contrast to the one-pion or the two-pion decays for which everything in the amplitude (29) or (34) is known, the matrix element

$\langle 3\,\pi \,|\, A^\mu \,|\, 0\rangle$ for the three-pion mode is more complicated and poorly known. Its most general expression depends on three form factors H_1, H_2, H_3, each of which is a function of three kinematical variables q^2, s_1, s_2 defined below, and consequently the decay rate calculation is rather model dependent. The three-pion decay amplitude is given by

$$\mathcal{M} = \frac{G_\mathrm{F}}{\sqrt{2}} V_\mathrm{ud}\overline{u}(k)\gamma_\mu(1-\gamma_5)u(P)\ \langle \pi_1(p_1)\,,\,\pi_2(p_2)\,,\,\pi_3(p_3)\,|\,A^\mu\,|\,0\rangle\ ,$$

$$\langle \pi_1(p_1)\,,\,\pi_2(p_2)\,,\,\pi_3(p_3)\,|\,A^\mu\,|\,0\rangle = \sum_{i=1,2,3} H_i(q^2, s_1, s_2)\frac{p_i^\mu}{m_\pi}\ ,$$

$$q^2 = (p_1+p_2+p_3)^2\ ,\ s_1 = (p_2+p_3)^2\ ,\ s_2 = (p_1+p_3)^2\ . \tag{13.53}$$

The existence of three form factors is easy to understand, since the only degrees of freedom for spinless pions are their momenta. The most general covariant structure of $\langle 3\,\pi \,|\, A^\mu \,|\, 0\rangle$ can be expressed in terms of these three independent momenta p_i^μ taken as a basis, their coefficients are the form factors H_i. Kinematically, each of these form factors is considered as a *four-point function* which connects the off-mass-shell W gauge boson to the three on-shell pions. The four-point function depends on q^2 (the virtual mass squared of the W) as well as on two independent Mandelstam variables that can be chosen as s_1 and s_2. These form factors $H_i(q^2, s_1, s_2)$ are not well determined, unlike the more familiar $F_\pi(q^2)$ in the two-pion case.

To compute the three-pion rate, the first approximation consists of considering the 3π as a quasi-two-body state $\rho(770) + \pi$ followed by $\rho \to 2\,\pi$. Then the matrix element may be written as

$$\langle \rho(p), \pi(p')\,|\,A^\mu\,|\,0\rangle = m_\rho\,\varepsilon^\mu K_1(q^2) + \frac{\varepsilon\cdot q}{m_\rho}\left[(p-p')^\mu K_2(q^2) + q^\mu K_3(q^2)\right]$$

where ε^μ is the four-vector polarization of the ρ and the three dimensionless form factors $K_i(q^2)$ depend only on $q^2 = (p+p')^2 = (p_1+p_2+p_3)^2$. These form factors $K_i(q^2)$ may be considered as linear combinations of the $H_i(q^2, s_1, s_2)$ in the limit where two pions form a ρ resonance. Then from the ρ propagator, the dependence of $H_i(q^2, s_1, s_2)$ on the variables $s_{j=1,2}$ would take the Breit–Wigner shape similar to (40), i.e. the s_j are concentrated near m_ρ^2 as $\left[m_\rho^2 - s_j - \mathrm{i}\sqrt{s_j}\Gamma_\rho(s_j)\right]^{-1}$ (Fig. 13.6).

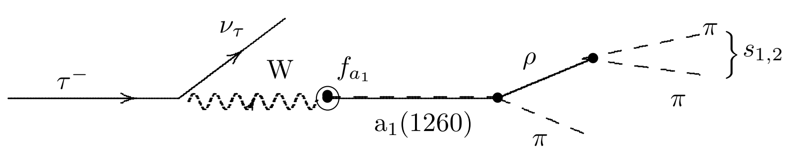


Fig. 13.6. $\tau^- \to \nu_\tau + \rho + \pi$ followed by $\rho \to 2\pi$

The next approximation is to assume the dominance of two resonances, the axial $a_1(1260)$ meson ($J^P = 1^+$) and the pseudoscalar $\pi(1300) \equiv \pi'$ meson. In analogy with the decay $\tau \to \nu_\tau + \rho$ followed by $\rho \to 2\,\pi$ considered previously, the three-pion mode may be approximated by $\tau \to \nu_\tau + a_1(1260)$ and $\tau \to \nu_\tau + \pi'$ followed by $a_1(1260) \to 3\pi$ and $\pi' \to 3\pi$. The spectral functions

$$a_1^{3\pi}(q^2) = 2\pi\, f_{a_1}^2\, \delta(q^2 - m_{a_1}^2) \ , \ \ a_0^{3\pi}(q^2) = 2\pi\, f_{\pi'}^2\, \delta(q^2 - m_{\pi'}^2) \ , \qquad (13.54)$$

correspond to the zero-width approximation of both $a_1(1260)$ and π', where the decay constants f_{a_1} and $f_{\pi'}$ of $a_1(1260)$ and π' are defined by

$$\big\langle a_1^-(1260)\,|\,A^\mu\,|\,0\big\rangle = m_{a_1}\, f_{a_1}\varepsilon^\mu \quad , \quad \langle \pi'(p)\,|\,A^\mu\,|\,0\rangle = \mathrm{i}\, f_{\pi'} p^\mu \ .$$

Unlike f_π and f_ρ, the $f_{\pi'}$ and f_{a_1} are not well determined experimentally. Weinberg sum rules[5] may be useful for an estimate of f_{a_1}. In our notation, the first Weinberg sum rule is written as

$$f_{\rho^\pm}^2 - f_{a_1}^2 = f_\pi^2 \ , \text{ where } \ f_{\rho^\pm} = \sqrt{2} f_\rho \ . \qquad (13.55)$$

The crudest estimation consists in keeping only $a_1^{3\pi}(q^2)$ and neglecting $a_0^{3\pi}(q^2)$. Using (49), the three-pion rate is given by

$$\begin{aligned}\frac{\Gamma(\tau^- \to \nu_\tau + 3\pi)}{\Gamma(\tau^- \to \nu_\tau + 2\pi)} &= \frac{\Gamma(\tau^- \to \nu_\tau + a_1^-)}{\Gamma(\tau^- \to \nu_\tau + \rho^-)} \\ &= \frac{f_{a_1}^2}{f_{\rho^\pm}^2}\left(\frac{M^2 - m_{a_1}^2}{M^2 - m_\rho^2}\right)^2 \frac{M^2 + 2m_{a_1}^2}{M^2 + 2m_\rho^2} \ . \end{aligned} \qquad (13.56)$$

Combining (56) with (55), the branching ratio $\mathrm{Br}(\tau^- \to \nu_\tau + 3\pi)$ is found to be $\approx 9\%$, lower than the data by a factor of 2. Since both the ρ and the $a_1(1260)$ are broad resonances, this zero-width approximation is rather poor. One may improve the estimation by replacing $\delta(q^2 - m_{a_1}^2)$ with a Breit–Wigner factor, in analogy with the ρ case discussed in (41) where the q^2 dependence of the width is taken into account. We write

$$\delta(q^2 - m_{a_1}^2) \longrightarrow \frac{\Gamma_{a_1}(q^2)\sqrt{q^2}}{\pi}\ \frac{1}{(q^2 - m_{a_1}^2)^2 + q^2\,\Gamma_{a_1}^2(q^2)} \ ,$$

$$\Gamma_{a_1}(q^2) = \Gamma_{a_1}\frac{m_{a_1}^2}{q^2}\left(\frac{q^2 - 9\,m_\pi^2}{m_{a_1}^2 - 9\,m_\pi^2}\right)^{3/2} \ , \text{ where } \ \Gamma_{a_1} \sim 400 \text{ MeV} \ ,$$

$$a_1^{3\pi}(q^2) = \frac{2\,f_{a_1}^2\,\Gamma_{a_1}(q^2)\sqrt{q^2}}{(q^2 - m_{a_1}^2)^2 + q^2\,\Gamma_{a_1}^2(q^2)} \ .$$

One may use this formula for $a_1^{3\pi}(q^2)$ and plug it into (49) to extract the decay constant f_{a_1} from the experimental branching ratio. The result is encouraging and gives $f_{a_1} \approx 250$ MeV.

[5] S. Weinberg, Phys. Rev. Lett. **18** (1967) 507

13.4.2 Spectral Functions of Quark Pairs

We calculated in the previous sections many exclusive modes $\tau^- \to \nu_\tau + H^-$, where H^- are π^-, K^-, $\pi^- + \pi^0$, (including ρ^-, K^{*-}), $\mathrm{K}^- + \overline{\mathrm{K}}^0$, 3π. The sum of these dominant decays almost saturates the observed semileptonic width, but still leaves out many other channels. We would like however to sum over all the exclusive semileptonic rates, which is, by definition, the inclusive decay rate $\Gamma(\tau^- \to \nu_\tau +$ hadrons). Doing such a sum, i.e. computing the whole spectral functions $v_{1,0}(q^2)$ and $a_{1,0}(q^2)$ for several particles, is not only cumbersome but also involves large uncertainties due to our lack of knowledge of the involved form factors. We must look for another approach.

Fortunately, the energy released by the heavy τ is large enough (on the QCD scale) that the quark-parton model is presumably valid. Then the inclusive rate may be saturated by $\tau^- \to \nu_\tau + \mathrm{d} + \overline{\mathrm{u}}$ and $\tau^- \to \nu_\tau + \mathrm{s} + \overline{\mathrm{u}}$. This saturation has the name of quark-hadron duality. For the calculation of these two rates, we need the spectral functions of quark pairs which replace those of all the exclusive hadronic states.

Let us denote the pair carrying the same color j (since it is created from the colorless $\mathrm{W}^\pm$) by $\mathrm{q}_2 + \overline{\mathrm{q}}_3$ with momentum p_2, p_3 and mass m_2, m_3 respectively. Defined in (46), the hadronic tensor $\mathcal{H}^{\mu\nu}_{\mathrm{q}_2\mathrm{q}_3}$ associated with the quark pair $\mathrm{q}_2+\overline{\mathrm{q}}_3$ in $\tau^-(P) \to \nu_\tau(k) + \mathrm{q}_2(p_2) + \overline{\mathrm{q}}_3(p_3)$ is

$$\mathcal{H}^{\mu\nu}_{\mathrm{q}_2\mathrm{q}_3} = |V_{\mathrm{q}_2\mathrm{q}_3}|^2 \int \frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_3}{2E_3}\frac{\delta^4(p_2+p_3-q)}{(2\pi)^2}\, 8\,[p_2^\mu\, p_3^\nu + p_2^\nu\, p_3^\mu - g^{\mu\nu}\,(p_2\cdot p_3)]\ ;$$

the last factor $8\ [p_2^\mu\, p_3^\nu + p_2^\nu\, p_3^\mu - g^{\mu\nu}\,(p_2\cdot p_3)]$ comes from the trace of $|\overline{u}(p_2)\gamma^\mu(1-\gamma_5)v(p_3)|^2$. In fact, the latter is a sum of a vectorial and an axial parts, which are given respectively by $4\,[p_2^\mu\, p_3^\nu + p_2^\nu\, p_3^\mu - g^{\mu\nu}\,(p_2\cdot p_3 \pm m_2\, m_3)]$. Their interference $8\,\mathrm{i}\,\varepsilon^{\mu\nu\alpha\beta}(p_2)_\alpha(p_3)_\beta$ does not contribute, since the integration is symmetric in $p_2,\ p_3$. We calculate the integration of the vectorial part first. With the implicit factor $|V_{\mathrm{q}_2\mathrm{q}_3}|^2/\pi^2$, the following quantity is to be evaluated:

$$\int \frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_3}{2E_3}\,\delta^4(p_2+p_3-q)\,[p_2^\mu\, p_3^\nu + p_2^\nu\, p_3^\mu - g^{\mu\nu}\,(p_2\cdot p_3 + m_2\, m_3)]\ . \quad (13.57)$$

This equation has the general form $-A\,q^2\,g^{\mu\nu} + B\,q^\mu q^\nu$ which can be rewritten as $(-q^2\,g^{\mu\nu} + q^\mu q^\nu)\,A + q^\mu q^\nu\,(B-A)$. Multiplying respectively (57) by $g_{\mu\nu}$ and $q_\mu q_\nu$, we get two equations for two unknowns A and B:

$$q^2\,(B-4A) = [-q^2 + m_2^2 + m_3^2 - 4m_2 m_3]\int \frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_3}{2E_3}\,\delta^4(p_2+p_3-q)\ ,$$

$$q^4(B-A) = \frac{(m_2-m_3)^2\,[q^2-(m_2+m_3)^2]}{2}\int \frac{\mathrm{d}^3 p_2}{2E_2}\frac{\mathrm{d}^3 p_3}{2E_3}\,\delta^4(p_2+p_3-q)\ .$$

Using (30) for the double integral of the above equation, the analytic expressions of A and B are obtained. Comparing with (47), we can identify the term A as $v_1(q^2)$ and $(B-A)$ as $v_0(q^2)$ (modulo the factor $|V_{\mathrm{q_2q_3}}|^2/\pi^2$). For each color, the spectral functions of the quark pair $\mathrm{q}_2+\overline{\mathrm{q}}_3$ are found to be

$$v_1(q^2) = \frac{|V_{\mathrm{q_2q_3}}|^2}{12\,\pi}\frac{\sqrt{\lambda(q^2,m_2^2,m_3^2)}}{q^2}\left\{2-\frac{m_2^2+m_3^2-6\,m_2\,m_3}{q^2}-\frac{(m_2^2-m_3^2)^2}{q^4}\right\}\,,$$

$$v_0(q^2) = \frac{|V_{\mathrm{q_2q_3}}|^2}{4\,\pi}\frac{\sqrt{\lambda(q^2,m_2^2,m_3^2)}}{q^2}\,\left\{\frac{(m_2-m_3)^2}{q^2}-\frac{(m_2^2-m_3^2)^2}{q^4}\right\}\,. \quad (13.58)$$

If $m_2 = m_3$, the vector current $\overline{q}_2\gamma^\mu q_3$ is conserved, and as expected, we get $v_0(q^2)=0$. The multiplicative color factor $N_{\mathrm{c}}=3$ must be included on the right-hand side of (58), since we sum over the quark colors. The $a_1(q^2)$ $[a_0(q^2)]$ can be deduced respectively from $v_1(q^2)$ $[v_0(q^2)]$ by changing only $m_3\to -m_3$ in (58). We get

$$\rho_1(q^2)\equiv v_1(q^2)+a_1(q^2) = \frac{C(q^2)}{6\,\pi}\left\{2-\frac{m_2^2+m_3^2}{q^2}-\frac{(m_2^2-m_3^2)^2}{q^4}\right\}\,,$$

$$\rho_0(q^2)\equiv v_0(q^2)+a_0(q^2) = \frac{C(q^2)}{2\pi}\left\{\frac{m_2^2+m_3^2}{q^2}-\frac{(m_2^2-m_3^2)^2}{q^4}\right\}\,, \quad (13.59)$$

where $C(q^2)\equiv N_{\mathrm{c}}\;|V_{\mathrm{q_2q_3}}|^2\;\sqrt{\lambda(q^2,m_2^2,m_3^2)}/q^2$. Putting (59) into (49) or (50) accordingly, the width $\Gamma(\tau^-\to\nu_\tau+\mathrm{q}_2+\overline{\mathrm{q}}_3)$ is given by

$$\begin{aligned}\Gamma =& N_{\mathrm{c}}\,\frac{G_{\mathrm{F}}^2\,|V_{\mathrm{q_2q_3}}|^2}{192\pi^3M^3}\int_{(m_2+m_3)^2}^{M^2}\mathrm{d}q^2(M^2-q^2)^2\frac{\sqrt{\lambda(q^2,m_2^2,m_3^2)}}{q^2}\\ &\times\left[(M^2+2q^2)\left(2-\frac{\sigma}{q^2}-\frac{\delta^2}{q^4}\right)+3M^2\left(\frac{\sigma}{q^2}-\frac{\delta^2}{q^4}\right)\right]\,,\\ \sigma\equiv& m_2^2+m_3^2\,,\ \delta\equiv m_2^2-m_3^2\quad,\qquad \lim_{m_2,m_3\to 0}\Gamma = N_{\mathrm{c}}\,|V_{\mathrm{q_2q_3}}|^2\,\Gamma_0\,. \quad (13.60)\end{aligned}$$

Using this formula, we compute the Cabibbo-favored decay width $\Gamma^{S=0}\equiv\Gamma(\tau^-\to\nu_\tau+\mathrm{d}+\overline{\mathrm{u}})$ and the Cabibbo-suppressed $\Gamma^{S=-1}\equiv\Gamma(\tau^-\to\nu_\tau+\mathrm{s}+\overline{\mathrm{u}})$. Their sum saturates the inclusive semileptonic width $\Gamma(\tau^-\to\nu_\tau+\mathrm{hadrons})$.

For $\Gamma^{S=0}$, we take $m_{\mathrm{d}}=m_{\mathrm{u}}=m$ and define $t=q^2/M^2$, $\eta=m^2/M^2$. The result is

$$\begin{aligned}\Gamma^{S=0} &= 2\,N_{\mathrm{c}}\,|V_{\mathrm{ud}}|^2\,\Gamma_0\int_{4\,\eta}^1\mathrm{d}t\,(1-t)^2\sqrt{1-4\frac{\eta}{t}}\left\{(1+2t)\left(1-\frac{\eta}{t}\right)+3\,\frac{\eta}{t}\right\},\\ &= N_{\mathrm{c}}\,|V_{\mathrm{ud}}|^2\,\Gamma_0\,G\left(\frac{m^2}{M^2},\frac{m^2}{M^2}\right)\,, \quad (13.61)\end{aligned}$$

$$G(x,x) = \sqrt{1-4x}\left[1-14x-2x^2-12x^3\right]+24\,x^2\,(1-x^2)\log\frac{1+\sqrt{1-4x}}{1-\sqrt{1-4x}}\,.$$

The $\Gamma^{S=-1}$ can be obtained similarly. Here the two masses $m_2 = m_\mathrm{s}$ and $m_3 - m_\mathrm{u}$ are unequal, the phase space integration in (60) is more involved,

$$\Gamma^{S=-1} = N_\mathrm{c}\,|V_\mathrm{us}|^2\,\Gamma_0\,G\left(\frac{m_2^2}{M^2},\frac{m_3^2}{M^2}\right)\ , \text{ where}$$

$$G(x,y) = \sqrt{\lambda(1,x,y)}\left[1-(x+y)(7+6xy)-(x^2+y^2)+(x-y)^2(x+y-6)\right] + 12\left[x^2(1-y^2)\log\frac{1+x-y+\sqrt{\lambda(1,x,y)}}{1+x-y-\sqrt{\lambda(1,x,y)}}+(x\leftrightarrow y)\right]\ . \tag{13.62}$$

Note that $G(x,y) = G(y,x)$. When x or y vanishes, (22) is recovered:

$$G(x,0) = G(0,x) = f(x) = 1-8x+8x^3-x^4-12x^2\log x\ .$$

The phase space factor (62) multiplied by Γ_0 is the width of a fermion F of mass M decaying into three other fermions $\mathrm{F}\to\mathrm{f}_1+\mathrm{f}_2+\overline{\mathrm{f}}_3$, when one of the three final fermions is massless. If the three fermions are massive with masses $m_k \neq 0\ , k=1,2,3$, the width is obtained by combining (46), (48), and (59). Thus $\Gamma(\mathrm{F}\to\mathrm{f}_1+\mathrm{f}_2+\overline{f}_3) = \Gamma_0\,I(\eta_1,\eta_2,\eta_3)$, with $\eta_k = m_k^2/M^2$:

$$\begin{aligned} I(\eta_1,\eta_2,\eta_3) =& \frac{1}{M^8}\int_{(m_2+m_3)^2}^{(M-m_1)^2}\frac{\mathrm{d}q^2}{q^2}\sqrt{\lambda(M^2,m_1^2,q^2)}\,\sqrt{\lambda(q^2,m_2^2,m_3^2)}\\ &\times\left[\left(2-\frac{\sigma}{q^2}-\frac{\delta^2}{q^4}\right)\Phi_1(q^2)+3\left(\frac{\sigma}{q^2}-\frac{\delta^2}{q^4}\right)\Phi_0(q^2)\right]\\ \Phi_1(q^2) =& (M^2-q^2)(M^2+2q^2)-m_1^2\,(2M^2-q^2-m_1^2)\ ,\\ \Phi_0(q^2) =& (M^2-q^2)M^2-m_1^2\,(2M^2+q^2-m_1^2)\ . \end{aligned} \tag{13.63}$$

This phase space factor $I(\eta_1,\eta_2,\eta_3)$ is *totally symmetric* in the permutation of the three arguments (Problem 13.8). Of course, $I(0,x,y) = G(x,y)$, $I(0,x,x) = G(x,x)$, $I(0,0,x) = f(x)$, and $I(0,0,0) = 1$.

The formula for $I(\eta_1,\eta_2,\eta_3)$ is particularly important when we study the heavy–flavored D and B meson decays (Chap. 16). In the limit $m_k = 0$ for all k, we recover $\Gamma^{S=0} = N_\mathrm{c}\,|V_\mathrm{ud}|^2\,\Gamma_0$, and $\Gamma^{S=-1} = N_\mathrm{c}\,|V_\mathrm{us}|^2\,\Gamma_0$, thus

$$\frac{\Gamma(\tau^-\to\nu_\tau+\text{hadrons})}{\Gamma(\tau^-\to\nu_\tau+\mathrm{e}^-+\overline{\nu}_\mathrm{e})} = N_\mathrm{c}\,(|V_\mathrm{ud}|^2+|V_\mathrm{us}|^2)\approx N_c\ . \tag{13.64}$$

The relation (64), like the ratio R defined in (7.132), may also be derived by the color-counting argument (5). We have (see also Sect. 7.5),

$$R = \frac{\sigma(\mathrm{e}^++\mathrm{e}^-\to\text{hadrons})}{\sigma(\mathrm{e}^++\mathrm{e}^-\to\mu^++\mu^-)} = N_\mathrm{c}\sum_k Q_k^2\ , \tag{13.65}$$

where Q_k is the charge (in units of $e>0$) of the quark k.

We close this section with a remark. First, we consider many exclusive hadronic channels of the τ decay, their rates are given by (31),(39), (56) in which enter parameters taken from elsewhere, like f_π, f_ρ, f_{a_1} and $\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \pi^+ + \pi^-)$. These parameters are not at all directly related to the τ properties. When all of these exclusive decay rates are summed up, it is quite possible *a priori* that the sum exceeds the inclusive rate $N_\mathrm{c}\Gamma_0\, G(x,y)$ described by the quark picture, which would be disastrous. Remarkably, the sum approaches $N_\mathrm{c}\Gamma_0\, G(x,y)$ from below and nearly saturates $N_\mathrm{c}\Gamma_0\, G(x,y)$. These quantitatively correct results support the quark-hadron duality.

Problems

13.1 The forbidden mode $\pi^- + \eta$. Explain why $\Gamma(\tau^- \to \nu_\tau + \pi^- + \eta) \ll \Gamma(\tau^- \to \nu_\tau + \mathrm{K}^- + \eta) \ll \Gamma(\tau^- \to \nu_\tau + \pi^- + \pi^0)$. Also why $\Gamma(\eta \to 2\pi) \ll \Gamma(\eta \to 3\pi)$.

13.2 Angular distribution in $\tau^- \to \nu_\tau + \pi^-$. For a τ^- at rest, its neutrino and the π^- come out back-to-back. Since ν_τ has helicity -1, show that the π^- prefers to be emitted parallel to the spin direction $\boldsymbol{S}$ of the τ^-. This property is confirmed by the following angular distribution. Show that

$$\frac{\mathrm{d}\Gamma(\tau^\mp \to \nu + \pi^\mp)}{\mathrm{d}\cos\theta} = \frac{G_\mathrm{F}^2\,|V_\mathrm{ud}|^2}{32\pi} f_\pi^2\, M^3 \left(1 - \frac{m_\pi^2}{M^2}\right)^2 (1 \pm \cos\theta)\ ,$$

where θ is the angle between $\boldsymbol{S}$ and the three-momentum of the π. Notice the $\pm\cos\theta$ of the above equation is opposite to $\mp\cos\theta$ in (24). Draw diagrams similar to Fig. 13.4 to explain this change of sign.

13.3 Michel parameter ρ and Fierz rearrangement. Consider the weak decay of a fermion F into three massless fermions, $\mathrm{F} \to \mathrm{f}_1 + \mathrm{f}_2 + \overline{\mathrm{f}}_3$. In the standard model, its matrix element has the structure $(\mathrm{V}-\mathrm{A}) \times (\mathrm{V}-\mathrm{A})$. Let us write it as $K_{12}[(\mathrm{V}-\mathrm{A})\times(\mathrm{V}-\mathrm{A})] \equiv \overline{f}_1\gamma^\mu(1-\gamma_5)F\,\overline{f}_2\gamma_\mu(1-\gamma_5)f_3$, the Fermi coupling $G_\mathrm{F}/\sqrt{2}$ is omitted. Show that the amplitudes $K_{12}[(\mathrm{V}\mp\mathrm{A})\times(\mathrm{V}\mp\mathrm{A})]$, $K_{12}[\mathrm{V}\times\mathrm{V}]$, and $K_{12}[\mathrm{A}\times\mathrm{A}]$ give rise to $\rho = 3/4$, whereas $K_{12}[\mathrm{V}\times(\mathrm{V}\pm\mathrm{A})]$ and $K_{12}[\mathrm{A}\times(\mathrm{V}\pm\mathrm{A})]$ yield $\rho = 3/8$. For the amplitude $K_{12}[(\mathrm{V}\mp\mathrm{A})\times(\mathrm{V}\pm\mathrm{A})]$, one gets $\rho = 0$. Using the Fierz rearrangement formulas given in the Appendix, $K_{12}[(\mathrm{V}\mp\mathrm{A})\times(\mathrm{V}\pm\mathrm{A})] \leftrightarrow K_{21}[\mathrm{S}\times\mathrm{P}]$, we then again understand $\rho = 0$.

13.4 Experimental determination of f_{K^*}, $f_{\pi'}$, and f_{a_1}. Why are these decay constants difficult to be measured, contrary to f_π and f_ρ ?

13.5 Strong evidence against the S, P structure from π decay. Write the matrix element of $\pi \to \ell + \nu_\ell$ if the weak interaction is of the S, P types. Compute the rate and show that

$$\frac{\pi^- \to \mathrm{e}^- + \nu_\mathrm{e}}{\pi^- \to \mu^- + \nu_\mu} \sim 1\ , \quad \text{which is in strong disagreement with experiments .}$$

13.6 Neutron decay $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$**.** We recall from (12.52) that the matrix element of the weak current $V_{\mathrm{ud}}\overline{u}\gamma_\mu(1-\gamma_5)d$, sandwiched between the neutron and the proton, has four form factors. Explain why only two of them dominate the neutron β-decay matrix element. The other two could be neglected, which ones? Compute the decay rate, using $m_\mathrm{n} = 939.5656$ MeV, $m_\mathrm{p} = 938.2723$ MeV, $m_\mathrm{e} = 0.5109$ MeV, $|V_{\mathrm{ud}}| = 0.9736$. Compare it with the neutron lifetime 887 ± 2 s. Deduce $g_1(0)$.

13.7 W propagator effect. Derive the last factor of (28), which is $1 + \frac{3}{5}\frac{M^2}{M_\mathrm{W}^2} - 2\frac{m^2}{M_\mathrm{W}^2}$.

13.8 Decay rate of a fermion into three massive fermions. Let us write the matrix element of $\mathrm{F}(P) \to \mathrm{f}_1(p_1) + \mathrm{f}_2(p_2) + \overline{\mathrm{f}}_3(p_3)$ as

$$\mathcal{M} = \frac{G_\mathrm{F}}{\sqrt{2}}\overline{u}(p_1)\gamma_\mu(1-\gamma_5)u(P)\,\overline{u}(p_2)\gamma^\mu(1-\gamma_5)v(p_3)\ .$$

First compute $Y \equiv \frac{1}{2}\sum_{\mathrm{spin}}|\mathcal{M}|^2$, keeping all final-state masses $m_{1,2,3} \neq 0$. Express Y in terms of $s = (p_1+p_2)^2$ and M^2, m_k^2. Show that $\Gamma = \Gamma_0 J(\eta_1,\eta_2,\eta_3)$, $\eta_k = m_k^2/M^2$, where the phase space factor $J(\eta_1,\eta_2,\eta_3)$ is

$$J(\eta_1,\eta_2,\eta_3) = \frac{12}{M^8}\int_{(m_1+m_2)^2}^{(M-m_3)^2}\frac{\mathrm{d}s}{s}\sqrt{\lambda(M^2,m_3^2,s)}\,\sqrt{\lambda(s,m_1^2,m_2^2)} \times (s-m_1^2-m_2^2)\,(M^2+m_3^2-s)\ , \tag{13.66}$$

implying that $J(\eta_1,\eta_2,\eta_3)$ is symmetric in the permutation of m_1 and m_2. Compare $J(\eta_1,\eta_2,\eta_3)$ with $I(\eta_1,\eta_2,\eta_3)$ in (63). Show that $J(\eta_1,\eta_2,\eta_3)$ must be equal to $I(\eta_1,\eta_2,\eta_3)$. So, $J(x,y,z) = I(x,y,z)$ is totally symmetric by permutation of their three arguments. What happens if the interaction is of the $(\mathrm{V}+\mathrm{A})\times(\mathrm{V}-\mathrm{A})$ type ? Show that the integrated width is identical to that of the $(\mathrm{V}-\mathrm{A})\times(\mathrm{V}-\mathrm{A})$ type.

Suggestions for Further Reading

Parity violation in weak interactions and spin-momentum correlation:

Lee, T. D. and Yang, C. N., Phys. Rev. **104** (1956) 254

Wu, C. S. et al., Phys. Rev. **105** (1957) 1413

Michel parameters as probe of weak interaction structures:

Bouchiat, C. and Michel, L., Phys. Rev. **106** (1957) 170

Jarlskog, C., Nucl. Phys. **75** (1966) 659

Marshak, R. E., Riazuddin and Ryan, C. P., *Theory of Weak Interactions in Particle Physics*. Wiley-Interscience, New York 1969

Systematic analyses of τ decay:

Alemany, R., Davier, M. and Höcker, A., Eur. Phys. J. **C2** (1998) 123

Braaten, E., Narison, S. and Pich, A., Nucl. Phys. **B373** (1992) 581

Davier, M., in *Proc. Second Workshop on Tau Lepton Physics* (ed. Gan, K. K.). World Scientific, Singapore 1992

Perl, M. L., Reports Progress Phys. **55** (1992) 653

Tsai, Yung-Su, Phys. Rev. **D4** (1971) 2821