10 Electron–Nucleon Scattering

This chapter begins with an introduction to the notion of form factors and structure functions which play a central role in all electromagnetic and weak processes involving hadrons. As functions of the Lorentz-invariant momentum transfer q^2, form factors parameterize the interactive effects of the constituents of the hadrons. First, we give an intuitive physical interpretation of the electromagnetic form factor as the charge distribution of the hadron and associate its slope with the hadron size. Next, we look at the form factors of weak interaction. Their normalization and dependence on q^2 are also discussed. A brief survey is made of their analytic property through dispersion relations and pole dominance. The nucleon form factors can be measured by elastic lepton–nucleon scattering, and the physical meaning of each term in the Rosenbluth formula for the cross-section is explained in detail.

We recall that a particle (of four-momentum q_μ) is virtual or off-mass-shell if its invariant mass squared $q^2 = q_0^2 - |\boldsymbol{q}|^2$ is not necessarily equal to its true mass squared m^2, e.g. a virtual photon has $q^2 \neq 0$. The invariant mass of the virtual photon exchanged in electron–nucleon scattering can be varied by changing the energy and/or the angle of the scattered electron. The possibility of varying q^2 in deep inelastic scattering provides a powerful probe of the detailed structure of the nucleon, showing that quarks are pointlike constituents of matter. We introduce the Bjorken scaling law of the nucleon structure functions and its interpretation by Feynman with the quark–parton picture, which describes so well experimental data. Evidence for gluons as hadronic constituents insensitive to electroweak interactions is also given.

10.1 Electromagnetic and Weak Form Factors

In our present understanding, based on direct and indirect experimental data, there is every reason to believe that leptons and quarks – the fundamental constituents of matter – are structureless down to a distance scale of 10^{-16} cm, independently of their other properties. From the very light or even massless neutrinos to the top quark as heavy as the Au nucleus, all of these twelve fermionic constituents are assumed to be pointlike in spite of the huge differences in their masses.

On the other hand, mesons and baryons (hadrons) have structure. Their static and dynamic properties deduced from their production and decay modes, together with their spectra, all indicate that hadrons are effectively bound states of quarks (see Chap. 7). Like any composite object, the hadrons naturally carry complicated spatial structures and behave differently from pointlike leptons in their electromagnetic and weak interactions. For instance, the lepton–hadron cross-sections decrease rapidly with q^2, in sharp constrast with the lepton–lepton cross-sections.

This can be understood intuitively as follows: in lepton–lepton scattering, if a pointlike lepton is hit by a photon emitted from the other lepton, the only effect is that its momentum will be changed in a way consistent with energy-momentum conservation; the strength of the interaction and therefore the cross-section is insensitive to the momentum transfer q^2. On the other hand, in lepton–hadron scattering, because of the interaction between the constituents of the hadron and the photon emitted from the lepton, the strength of the interaction will depend on q^2; the more q^2 increases, the more the inner structure of the composite target can be probed and the nature of the interaction between the constituents can be revealed.

To describe the hadron structure, the standard approach is to introduce a *form factor*, which is the Fourier transform in momentum space of the spatial structure of the hadron. Its physical meaning is illustrated by the following example.

Let us consider the scattering of an electron by the static Coulomb field of a heavy nucleus (Rutherford experiment at the beginning of the century). In our contemporary language, the electron–nucleus interaction is governed by the exchange of a virtual spacelike photon between the projectile (electron) and the target (nucleus). When the three-momentum $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ transferred to the photon by the incoming electron $\boldsymbol{p}_i$ and the outgoing electron $\boldsymbol{p}_f$ is small, say $|\boldsymbol{q}| \sim 20\,\text{KeV} \sim 10^9/\text{cm} = 1/(10^{-9}\text{cm})$, the electromagnetic probe cannot penetrate the interior of the nucleus, which has a much smaller size $\sim 10^{-12}\,\text{cm}$, as if the latter were simply a pointlike positive charge Ze, $e > 0$. As the transferred momentum increases, say up to $20\,\text{MeV} = 1/(10^{-12}\,\text{cm})$ or higher, the complexity of the nucleus becomes more and more transparent and the photon starts to *see* the protons with their electric charges distributed inside the nucleus. The Coulomb potential of a *pointlike nucleus* should be replaced by that of an extended object

$$\frac{Ze}{4\pi\, r} \equiv \frac{Ze}{4\pi\, |\boldsymbol{x}|} \longrightarrow \int \mathrm{d}^3y \frac{\rho(\boldsymbol{y})}{4\pi\, |\boldsymbol{x}-\boldsymbol{y}|} \equiv \frac{1}{4\pi} V(r) \;, \tag{10.1}$$

where $\rho(\boldsymbol{y})$ is the charge density of protons inside the nucleus, normalized to Ze: $\int \rho(\boldsymbol{y})\mathrm{d}^3y = Ze$. An unrealistic, structureless nucleus may be considered as a special case where all the protonic charges are concentrated at a single point: $\rho(\boldsymbol{y}) = Ze\,\delta^3(\boldsymbol{y})$. The Fourier transforms in momentum space of the potentials $Ze/4\pi\, r$ and $V(r)/4\pi$ are denoted by $V_{\text{pt}}(\boldsymbol{q})$ and $V(\boldsymbol{q})$ respectively,

$$V_{\text{pt}}(\boldsymbol{q}) \equiv \frac{Ze}{4\pi} \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \frac{1}{r} \quad , \quad V(\boldsymbol{q}) \equiv \frac{1}{4\pi} \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} V(r) \;. \tag{10.2}$$

Our purpose is to show that $V(\boldsymbol{q})$ and $V_{\rm pt}(\boldsymbol{q})$ are related by a measurable nuclear form factor $F_{\rm N}(q^2)$ defined in (8) and (9). In (2), the three-dimension Fourier transformation proceeds only with a three-vector $\boldsymbol{q}$ adapted to the nonrelativistic case of a heavy nucleus of mass M considered here. The transferred energy $q_0 \equiv \sqrt{M^2+|\boldsymbol{q}|^2} - M$ is practically zero, only $\boldsymbol{q}$ enters, and the fourth-component Fourier transform $\int {\rm d}t\ {\rm e}^{{\rm i}q_0 t} = 2\pi\,\delta(q_0)$ simply refers to this fact. Let us first consider $V_{\rm pt}(\boldsymbol{q})$,

$$\begin{aligned}\int {\rm d}^3x\ {\rm e}^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{x}}\frac{1}{r} &= \lim_{\mu\to 0}\int {\rm d}^3x\ {\rm e}^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{x}}\frac{{\rm e}^{-\mu r}}{r} \\ &= \lim_{\mu\to 0}\ 2\pi\int_0^\infty \frac{{\rm e}^{-\mu r}}{r}r^2{\rm d}r\int_{-1}^{1}{\rm e}^{-{\rm i}|\boldsymbol{q}|\cdot r\cos\theta}{\rm d}\cos\theta \\ &= \lim_{\mu\to 0}\frac{4\pi}{(|\boldsymbol{q}|^2+\mu^2)} = \frac{4\pi}{|\boldsymbol{q}|^2}\ . \end{aligned} \tag{10.3}$$

The parameter μ (which has the dimension of mass or inverse of length, since μr is dimensionless) is introduced to make the integral easier to handle, the final result is however independent of it. We have

$$V_{\rm pt}(\boldsymbol{q}) = \frac{Ze}{|\boldsymbol{q}|^2}\ . \tag{10.4}$$

Instead of (4), where everything is expressed in terms of the three-vector $\boldsymbol{q}$, it would be nice to have a covariant form with the Lorentz-invariant $q^2 = q_0^2 - |\boldsymbol{q}|^2$ of the four-vector q_μ. The Fourier transform of the static Coulomb potential corresponds, as we have seen, to zero energy transfer for which the invariant q^2 takes the $-|\boldsymbol{q}|^2$ value. One then naturally arrives at the prescription: $|\boldsymbol{q}|^2$ is to be replaced by $-q^2$. Note that the invariant q^2 can be timelike ($q^2 > 0$) or spacelike ($q^2 < 0$), however in all scattering processes, q^2 is necessarily spacelike and the substitution $q^2 \to -|\boldsymbol{q}|^2 < 0$ is natural. We rewrite (4) in the form

$$V_{\rm pt}(q^2) = \frac{-Ze}{q^2}\ . \tag{10.5}$$

Yukawa potential. We can make a remark about (3): from (2) and (4), the nonrelativistic limit $1/|\boldsymbol{q}|^2$ of the propagator $-1/q^2$ of a massless boson mediated between the electron and the nucleus gives rise to a potential proportional to $1/r$.

By the same trick, when going back from the bottom to the top right-hand side of (3), we realize that the nonrelativistic limit $1/(|\boldsymbol{q}|^2+\mu^2)$ of $-1/(q^2-\mu^2)$ (propagator of an exchanged spinless particle of mass μ) can generate the potential ${\rm e}^{-\mu r}/r$. The range of the force is $1/\mu$ (since for a distance beyond this range, $\mu r > 1$, ${\rm e}^{-\mu r}$ becomes exponentially negligible).

It results that the exchange of a meson through its nonrelativistic propagator is the source of the interacting potential between two particles

$$\int \frac{\mathrm{d}^3 q\ \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{(2\pi)^3} \frac{1}{|\boldsymbol{q}|^2} = \frac{1}{4\pi\, r} \quad ; \quad \int \frac{\mathrm{d}^3 q\ \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{(2\pi)^3} \frac{1}{|\boldsymbol{q}|^2 + \mu^2} = \frac{\mathrm{e}^{-\mu r}}{4\pi\, r} \quad .$$

This fundamental concept that brings together the two-body potential $\mathrm{e}^{-\mu r}/r$ and the mass μ of their mediated meson was discovered by Yukawa in 1935. Its physical meaning is already mentioned in Chap. 1 (Fig. 1.1). Knowing the nuclear force range of about 1 or 2 fm [1 fm $= 10^{-13}$cm $\approx (200\,\mathrm{MeV})^{-1}$], Yukawa then predicted the existence and the mass between 100–200 MeV of a spinless particle, which later turns out to be the π meson exchanged between nucleons. When $\mu \to 0$, we recover the Coulomb potential; the infinite range of the electromagnetic force is a direct consequence of massless photons. Similarly, if the Coulomb potential between two charges e_1 and e_2 is $e_1 e_2/4\pi r$, the nuclear potential produced by the π meson exchanged between the two nucleons would be $g^2_{\pi\mathrm{NN}}\mathrm{e}^{-r\, m_\pi}/4\pi r$, where $g_{\pi\mathrm{NN}}$ is the pion–nucleon coupling constant ($g^2_{\pi\mathrm{NN}}/4\pi \approx 13.5$). The spin effect of the nucleon can also be incorporated and yields the one-pion-exchange (OPE) nucleon–nucleon force (Problem 10.1). It is important not to confuse the notion of form factors considered here with the Yukawa mechanism that provides the interacting potential $\mathrm{e}^{-\mu r}/r$ from an exchanged boson of mass μ. ■

Let us go back to the potential $V(\boldsymbol{q})$ in (1) and (2):

$$V(\boldsymbol{q}) = \frac{1}{4\pi} \int \mathrm{d}^3 x\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \int \mathrm{d}^3 y \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \ . \tag{10.6}$$

Putting $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{z}$ and using (3), we obtain

$$V(\boldsymbol{q}) = \frac{1}{4\pi} \int \mathrm{d}^3 z \frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z}}}{|\boldsymbol{z}|} \int \mathrm{d}^3 y\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} \rho(\boldsymbol{y}) = \frac{F_{\mathrm{N}}(q^2)}{|\boldsymbol{q}|^2} \ . \tag{10.7}$$

$V(\boldsymbol{q})$ is the nonrelativistic version of $V(q^2)$ and $F_{\mathrm{N}}(q^2)$ defined by

$$F_{\mathrm{N}}(q^2 = -|\boldsymbol{q}|^2) \equiv \int \mathrm{d}^3 y\ \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} \rho(\boldsymbol{y}) \tag{10.8}$$

is the Fourier transform of $\rho(\boldsymbol{y})$, the proton charge distribution in the nucleus. $F_{\mathrm{N}}(q^2)$ is called the electromagnetic form factor of the nucleus, with the normalization $F_{\mathrm{N}}(0) = Ze$, as can be seen by putting $|\boldsymbol{q}| = 0$ in (8) and remembering that $\int \mathrm{d}^3 y\, \rho(\boldsymbol{y}) = Ze$. From (4) to (7) we get

$$V(q^2) = V_{\mathrm{pt}}(q^2) \frac{F_{\mathrm{N}}(q^2)}{F_{\mathrm{N}}(0)} = \frac{-F_{\mathrm{N}}(q^2)}{q^2} \ . \tag{10.9}$$

The meaning of form factors can be seen by comparing (5) with (9).

Now in (8) we expand $\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} = 1 - \mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{y} - \frac{1}{2}|\boldsymbol{q}|^2 r^2\cos^2\theta + \cdots$, then after the integration over d^3y, we obtain $F_\mathrm{N}(q^2) = F_\mathrm{N}(0)[1 + \frac{1}{6}\langle r^2\rangle q^2 + \cdots]$ (remember the $|\boldsymbol{q}|^2 \to -q^2$ substitution). The integration of the linear term $\boldsymbol{q}\cdot\boldsymbol{y}$ vanishes by the spatial symmetry, and the coefficient $1/6 = (1/2)\ (1/3)$ comes from averaging $\cos^2\theta$. The quantity

$$\langle r^2\rangle \equiv \frac{6}{F_\mathrm{N}(0)}\left|\frac{\mathrm{d}F_\mathrm{N}(q^2)}{\mathrm{d}q^2}\right|_{q^2=0}$$

represents the squared radius of the nucleus.

Equations (8) and (9) show that the notion of form factor is appropriate for describing the particle structure, the slope of the form factor at $q^2 = 0$ gives the hadron size; *the greater the object is, the faster its form factor decreases*. The electromagnetic and weak form factors of a pointlike object are q^2-independent. Form factors are directly measurable physical quantities. We will see later in (37) that by performing electron scattering on nuclei, we can measure $F_\mathrm{N}(q^2)$ via the Rutherford or Mott cross-section, and once $F_\mathrm{N}(q^2)$ is obtained, by the inverse Fourier transformation of (8), we get $\rho(\boldsymbol{y})$, the distribution of protons inside the nucleus:

$$\rho(\boldsymbol{y}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3q\,\mathrm{e}^{+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{y}} F_\mathrm{N}(q^2 = -|\boldsymbol{q}|^2)\ .$$

The notion of the nucleus form factor $F_\mathrm{N}(q^2)$, taken as an illustrative example, can be generalized to hadrons. The electromagnetic or weak properties of the latter are influenced by strong interactions of their constituents (quarks and gluons) which in turn provide them with form factors, in the same way as the protons – constituents of nuclei – induce the nucleus form factor $F_\mathrm{N}(q^2)$.

For example, the $\pi^\pm$ and $\mathrm{K}^\pm$ mesons are not pointlike, their radii can be measured by $\mathrm{e}^+ + \mathrm{e}^- \to \pi^+ + \pi^-$ or $\mathrm{K}^+ + \mathrm{K}^-$ (Fig. 10.1).

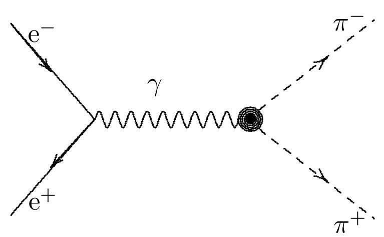

Fig. 10.1. Measurement of the pion form factor $F_\pi(q^2)$ and its radius $\langle r_\pi\rangle$ by $\mathrm{e}^+ + \mathrm{e}^- \to \pi^+ + \pi^-$

The *anomalous* magnetic moment of the proton is another manifestation of the proton structure. In the following Gordon decomposition of a pointlike fermionic current $\overline{u}(P')\gamma_\mu u(P)$ (see the Appendix),

$$e\overline{u}(P')\gamma_\mu u(P)A^\mu = \frac{e}{2M}\overline{u}(P')\left[(P'+P)_\mu + \mathrm{i}\sigma_{\mu\nu}(P'-P)^\nu\right]u(P)A^\mu\ , \quad (10.10)$$

the second term $(e/2M)\overline{u}(P')\,[\mathrm{i}\sigma_{\mu\nu}(P'-P)^\nu]\,u(P)A^\mu$ – which can be written in the nonrelativistic limit as $(e/2M)\overline{u}(P')\boldsymbol{\sigma}\cdot\boldsymbol{B}u(P)$ – clearly indicates that the magnetic moment of a pointlike fermion of charge e and mass M is equal to one Bohr magneton $\mu_\mathrm{B} \equiv (e/2M)$. Here $\boldsymbol{B}$ is the external magnetic field, and $\frac{1}{2}\boldsymbol{\sigma}$ is the proton spin.

Experimentally, the magnetic moment of the proton turns out to be 2.79 μ_B. The difference of 1.79 μ_B, called the *anomalous* magnetic moment, is a manifestation of the complex structure of the proton. Compared with the anomalous term of the pointlike electron which is very close to $\alpha_\mathrm{em}/2\pi = 0.001161$ [due to QED radiative corrections first computed by Schwinger, see (14.15)], the proton anomalous magnetic moment is very large indeed. Moreover, experiments show that the neutron also has a magnetic moment equal to $-1.91\,\mu_\mathrm{B}$ and not zero as naively expected for a neutral pointlike particle. As discussed in Sect. 7.4.3, these anomalous magnetic moments can be viewed as the effects of the quark constituents of the nucleon.

In brief, probed by electromagnetic (weak) currents, the electromagnetic (weak interaction) properties of hadrons can be described by form factors which encapsulate strong interaction effects of the hadronic constituents. Form factors are usually parameterized as $F(q^2) = F(0)/(1-\frac{q^2}{\Lambda^2})^n$; the powers $n = 1, 2$ correspond to monopole and diplole respectively, and Λ is the pole mass. Since electroweak cross-sections and decay rates of hadrons are functions of form factors, the importance of $F(0)$ and of the q^2 behavior is evident. In principle, the strong interaction dynamics of the quark and gluon constituents should determine the q^2 dependence and the value $F(0)$ of the hadronic form factors; however their determination is far from being achieved at present although there has been significant progress. The main reason is that we are in the low-energy QCD regime (whimsically called infrared slavery to contrast with ultraviolet asymptotic freedom corresponding to high energies) which deals with bound state problems for which the strong coupling constant is not small, and a perturbative treatment is inadequate. Nonperturbative methods, such as quark models, QCD sum rules, or lattice gauge theory, not discussed in this book, are still inconclusive at present.

However, even without any guidance from the dynamics of the hadronic constituents, the form factors can be obtained kinematically by using only a few principles: Lorentz invariance, conservation of currents, and *heavy flavor symmetry* (Chap. 16). In some cases, the normalizations are also fixed by these principles. We give in the following three typical examples: the pion and the nucleon electromagnetic form factors, and the weak form factors involved in the semileptonic decay $\overline{\mathrm{B}} \to \mathrm{D} + \ell^- + \overline{\nu}_\ell$ of the bottom B meson.

Example 10.1 Electromagnetic Pion Form Factor

The most general amplitude for the interaction of the charged pions $\pi^\pm$ (of initial four-momentum k and final four-momentum k') with the photon $\varepsilon_\mu(q)$ can be written as $\pm e\,\varepsilon_\mu(q)\,T^\mu(k',k)$, where

$$T^\mu(k',k) \equiv \langle \pi(k')\,|\,J^\mu_\mathrm{em}(0)\,|\,\pi(k)\rangle\,.$$

The quantity T^μ must obey $q_\mu T^\mu = 0$, $q_\mu = (k' - k)_\mu$ since the electromagnetic current J^μ_{em} is conserved, i.e. $\partial_\mu J^\mu_{\text{em}} = 0$. With spinless particles, we have at our disposal only their momenta k and k' as degrees of freedom, hence the most general Lorentz four-vector T^μ should have the form $a(k'+k)^\mu + bq^\mu$, which is reduced to $b = 0$ by $q_\mu T^\mu = bq^2 = 0$. Consequently, $(k'+k)^\mu F_\pi(q^2)$ is the only possible form of $T^\mu(k', k)$, and $F_\pi(q^2)$ is called the charged pion form factor, normalized by the condition $F_\pi(0) = 1$ (Problem 10.2),

$$\langle \pi(k') \,|\, J^\mu_{\text{em}}(0) \,|\, \pi(k) \rangle = (k'+k)^\mu F_\pi(q^2)\,, \qquad F_\pi(0) = 1\,. \tag{10.11}$$

For simplicity, we omit the standard one-particle state normalization factor $1/\left[(2\pi)^3\sqrt{4E_k E'_k}\right]$ in (11). The electromagnetic interaction of a pointlike spinless particle $\pm e\,\varepsilon_\mu(q)(k'+k)^\mu$ becomes $\pm e\,\varepsilon_\mu(q)(k'+k)^\mu F_\pi(q^2)$ for the $\pi^\pm$ meson

$$\begin{array}{ccc} \pm\, e\,(k'+k)^\mu & \longrightarrow & \pm e\,(k'+k)^\mu F_\pi(q^2)\,, \\ \text{pointlike pion} & \longrightarrow & \text{physical pion}\,. \end{array}$$

The above substitution is reminiscent of (9) for a spinless nucleus with its form factor $F_{\text{N}}(q^2)$. The proton distribution inside the nucleus is responsible of $F_{\text{N}}(q^2)$, similarly the strong interaction effects due to the quark and gluon constituents of the pion produce the form factor $F_\pi(q^2)$ (Fig. 10.2a). This form factor is already measured in $\text{e}^- + \text{e}^+ \to \pi^- + \pi^+$, where $q^2 \geq 4m_\pi^2$ is timelike (Problem 10.3). It may be also measured by the pion scattering on atomic electrons $\text{e}^- + \pi^\pm \to \text{e}^- + \pi^\pm$ or by pion electroproduction, in which $q^2 \leq 0$ is spacelike [see (36) below]. The size of the pion, $\sqrt{\langle r_\pi^2 \rangle}$, is given by

$$\langle r_\pi^2 \rangle = 6 \left| \frac{\text{d}F_\pi(q^2)}{\text{d}q^2} \right|_{q^2=0}\,.$$

Of course, all other charged spinless flavored mesons, such as $\text{K}^\pm$, $\text{D}^\pm$, and $\text{B}^\pm$ have similar electromagnetic form factors $F_{\text{K}}(q^2), F_{\text{D}}(q^2)$, and $F_{\text{B}}(q^2)$, normalized by $F_{\text{K,D,B}}(0) = 1$ as in the pion case. We emphasize that these normalizations are *model independent results*, due to the conserved electromagnetic current. These values at (and only at) $q^2 = 0$ are not modified by strong interaction effects of the hadronic constituents (quarks and gluons). This important result is known as the nonrenormalized form factors.

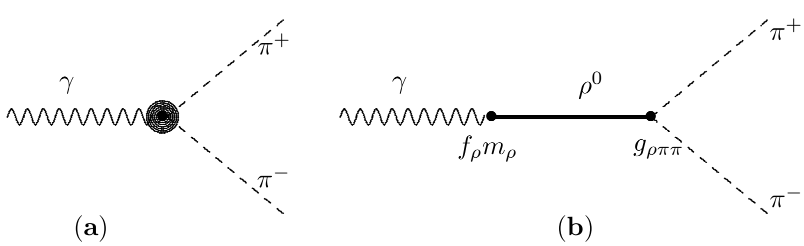

Fig. 10.2. (**a**) Pion form factor $F_\pi(q^2)$; (**b**) ρ^0 dominance of $F_\pi(q^2)$

Example 10.2 Electromagnetic Nucleon Form Factors
To describe the electromagnetic current $J^\mu_{\rm em}$ of the nucleon on the most general grounds, we note that there are in all four matrices of the vectorial type to be inserted between the nucleon spinors $\overline{u}(P')$ and $u(P)$. They are γ_μ , $\mathrm{i}\sigma_{\mu\nu}q^\nu$, $q_\mu \equiv (P'-P)_\mu$, and $(P'+P)_\mu$. However, from the Gordon decomposition (10), the term $(P'+P)_\mu$ can be written as a combination of γ_μ and $\mathrm{i}\sigma_{\mu\nu}q^\nu$; in addition, the conservation of the electromagnetic current implies that the term proportional to q_μ must be zero, using both $q_\mu\overline{u}(P')\gamma^\mu u(P) = 0$ and the antisymmetry of $\sigma_{\mu\nu}$. Therefore, the most general form of the nucleon electromagnetic current is expressed in terms of only two dimensionless form factors $F_1(q^2)$ and $F_2(q^2)$:

$$\begin{aligned} e\,\overline{u}(P')\gamma_\mu u(P) \quad &\longrightarrow \quad e\,\overline{u}(P')\left[\gamma_\mu F_1(q^2) + \mathrm{i}\frac{\sigma_{\mu\nu}q^\nu}{2M}F_2(q^2)\right]u(P)\,, \\ \text{pointlike nucleon} \quad &\longrightarrow \quad \text{physical nucleon}\ , \end{aligned} \tag{10.12}$$

with $F_1^{\rm p}(0) = 1$ (proton electric charge), $F_1^{\rm n}(0) = 0$ (neutron electric charge), $F_2^{\rm p}(0) = 1.79$ (proton anomalous magnetic moment), and $F_2^{\rm n}(0) = -1.91$ (neutron anomalous magnetic moment).

As in the case of meson form factors $F_{\rm M}(0) = 1$ (M stands for mesons), the normalizations $F_1^{\rm p}(0) = 1$, $F_1^{\rm n}(0) = 0$ are exact results due to the conserved electric current. On the other hand, the anomalous magnetic terms $F_2^{\rm p,n}(0)$ cannot be computed from first principle.

We may consider the proton and neutron as the $I_3 = +1/2$ and $I_3 = -1/2$ components of an isospin doublet $I = 1/2$, and define the isoscalar $F_i^0(q^2)$ and isovector $F_i^1(q^2)$ form factors by $(i = 1, 2)$:

$$F_i^0(q^2) = F_i^{\rm p}(q^2) + F_i^{\rm n}(q^2)\,; \qquad F_i^1(q^2) = F_i^{\rm p}(q^2) - F_i^{\rm n}(q^2)\,. \tag{10.13}$$

With $\frac{1}{2}(1+\tau_3)u = u_{\rm p}$, $\frac{1}{2}(1-\tau_3)u = u_{\rm n}$, (12) may be rewritten as

$$\frac{e}{2}\overline{u}(P')\left\{\left[\gamma_\mu F_1^0(q^2) + \mathrm{i}\frac{\sigma_{\mu\nu}q^\nu}{2M}F_2^0(q^2)\right] + \left[\gamma_\mu F_1^1(q^2) + \mathrm{i}\frac{\sigma_{\mu\nu}q^\nu}{2M}F_2^1(q^2)\right]\tau_3\right\}u(P).$$

These form factors can be measured by the elastic scattering $\mathrm{e}^- + \mathrm{N} \to \mathrm{e}^- + \mathrm{N}$ ($q^2 \leq 0$, see Fig. 4.9 and Sect. 10.3 below) or by the annihilation $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{N} + \overline{\mathrm{N}}$ ($q^2 \geq 4M^2$). The isovector form factors $F_1^1(q^2)$, $F_2^1(q^2)$ are useful when we study weak currents in nucleon β-decay together with the conserved vector current (CVC) property (Chap. 12). The same CVC relates the electromagnetic pion form factor $F_\pi(q^2)$ to the weak one, the latter appears for example in pion β-decay $\pi^- \to \pi^0 + \mathrm{e}^- + \overline{\nu}_{\rm e}$ (Problem 10.4).

Example 10.3 $\overline{\mathrm{B}} \to \mathrm{D}$ Weak Decay Form Factors
The amplitude $\mathcal{M}$ of the semileptonic decay $\overline{\mathrm{B}}(p) \to \mathrm{D}(p') + \ell^-(k_1) + \overline{\nu}_\ell(k_2)$ (the four-momenta p, p', k_1, k_2 of these particles are indicated in parentheses)

is a product of two matrix elements, that of the $\mathrm{V}-\mathrm{A}$ left-handed hadronic weak current H^μ taken between these meson states and that of the $\mathrm{V}-\mathrm{A}$ leptonic current $L_\mu = \overline{\ell}\gamma_\mu(1-\gamma_5)\nu_\ell$ taken between the vacuum and the lepton pair ℓ–$\overline{\nu}_\ell$, where $\langle \ell^-(k_1), \overline{\nu}_\ell(k_2) \,|\, L_\mu \,|\, 0\rangle = \overline{u}(k_1)\gamma_\mu(1-\gamma_5)v(k_2)$:

$$\mathcal{M} = \frac{G_\mathrm{F}}{\sqrt{2}} V_\mathrm{cb} \left\langle \ell^-(k_1), \overline{\nu}_\ell(k_2) \,|\, L_\mu \,|\, 0 \right\rangle \times \left\langle \mathrm{D}(p') \,|\, H^\mu \,|\, \overline{\mathrm{B}}(p) \right\rangle \ . \tag{10.14}$$

The hadronic current $H^\mu = \overline{c}\gamma^\mu(1-\gamma_5)b \equiv V^\mu - A^\mu$ is written in terms of the relevant charm and bottom quark fields, V_cb is the corresponding Cabibbo–Kobayashi–Maskawa (CKM) flavor mixing, and G_F is the Fermi coupling constant (Chap. 9). The second-quantized $b(x)$ field represents the annihilation operator of the b quark, while the $\overline{c}(x)$ field refers to the creation of the c quark, corresponding to the decay $\mathrm{b} \to \mathrm{c} + \ell^- + \overline{\nu}_\ell$ we are considering.

Since both $\overline{\mathrm{B}}(\mathrm{b\overline{q}})$ and $\mathrm{D}(\mathrm{c\overline{q}})$ are pseudoscalar ($J^P = 0^-$) mesons, one has $\left\langle \mathrm{D}(p') \,|\, A^\mu \,|\, \overline{\mathrm{B}}(p) \right\rangle = 0$ from general considerations of Lorentz covariance and parity. Indeed, with only two momenta p_α and p'_β as degrees of freedom at our disposal, there is no way to build up a matrix element of an axial-vector A^μ sandwiched between two spinless mesons having the *same intrinsic parity*. The matrix element of A^μ in this case must have the structure $\varepsilon^{\mu\alpha\beta\gamma} p_\alpha p'_\beta P_\gamma$ suited to its $J^P = 1^+$ property. But an independent third vector P_γ is lacking to construct such a term. So, $\langle 0^\pm \,|\, A^\mu \,|\, 0^\pm\rangle = 0$ for all $J^P = 0^\pm$ mesons.

There remains the vector part $\langle 0^\pm \,|\, V^\mu \,|\, 0^\pm\rangle \neq 0$. If one particle is pseudoscalar, the other is scalar (or the vacuum), then the roles of V^μ and A^μ are interchanged, i.e. $\langle 0^\pm \,|\, V^\mu \,|\, 0^\mp\rangle = 0$ while $\langle 0^\pm \,|\, A^\mu \,|\, 0^\mp\rangle \neq 0$. The best example is $\langle 0 \,|\, A^\mu \,|\, \pi(k)\rangle = \mathrm{i} f_\pi k^\mu$. From Lorentz covariance, the most general matrix element of V^μ sandwiched between the two $J^P = 0^-$ mesons is

$$\left\langle \mathrm{D}(p') \,|\, V^\mu \,|\, \overline{\mathrm{B}}(p) \right\rangle = f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu \ , \tag{10.15}$$

where $q \equiv p - p' = k_1 + k_2$ is the four-momentum transfer and $f_+(q^2)$ and $f_-(q^2)$ are the dimensionless weak transition $\overline{\mathrm{B}} \to \mathrm{D}$ form factors which are functions of the invariant q^2. Contrary to the electromagnetic current J^μ_em of charged pions considered in (11), the weak vector current $V^\mu = \overline{c}\gamma^\mu b$ is not conserved $[q_\mu V^\mu \propto (m_b - m_c) \neq 0]$, hence we have two form factors $f_\pm(q^2)$ instead of a single $F_\pi(q^2)$ as in the pion case. Here the timelike $q^2 = (k_1+k_2)^2$ represents also the squared invariant mass of the lepton pair, it varies within the range $m_\ell^2 \leq q^2 \leq (M_\mathrm{B} - M_\mathrm{D})^2 \equiv q^2_\mathrm{max}$.

Unlike the case of electromagnetic interactions with the exact results (11), the normalizations of the weak form factors are in general unknown. However, in the limit of infinitely heavy quark masses, $\Lambda_\mathrm{QCD} \ll M_\mathrm{B}, M_\mathrm{D} \to \infty$, a new *heavy flavor symmetry* (Chap. 16) appears in the effective Lagrangian of the standard model. This symmetry provides *model-independent* normalization of the weak form factors $f_\pm(q^2_\mathrm{max})$ at q^2_max, using a method similar to the derivation of $F_\pi(0) = 1$. The results are

$$f_+(q^2_\mathrm{max}) = \frac{M_\mathrm{B} + M_\mathrm{D}}{2\sqrt{M_\mathrm{B} M_\mathrm{D}}} \quad , \quad f_-(q^2_\mathrm{max}) = -\frac{M_\mathrm{B} - M_\mathrm{D}}{2\sqrt{M_\mathrm{B} M_\mathrm{D}}} \ . \tag{10.16}$$

The normalizations (16) are of great importance for the determination of V_{bc}. Its proof based on the heavy flavor symmetry will be given in (16.75).

10.2 Analyticity and Dispersion Relation

We remark that form factors are analytic functions in the complex q^2 plane except for singularities on the real timelike $q^2 \geq 0$ axis. This property is illustrated by a calculation of the magnetic form factor in Chap. 14; we also show, in another explicit example of the vacuum polarization (Chap. 15), that Feynman loop amplitudes are analytic. The singularities (poles or cuts) exist whenever the variable q^2 has values for which it is possible for all the particles in an intermediate state to be on the mass shell, i.e. to be physical [see Fig. 10.2b and (15.32)]. Let us take the simplest example of $F_\pi(q^2)$. For $q^2 \geq 4m_\pi^2$, the virtual photon can produce two on-mass-shell pions, $F_\pi(q^2)$ becomes a complex function with a cut starting at $q^2 \geq 4m_\pi^2$.

The discontinuity of $F_\pi(q^2)$ above and below the cut gives the imaginary part of $F_\pi(q^2)$: $F_\pi(q^2 + \mathrm{i}\varepsilon) - F_\pi(q^2 - \mathrm{i}\varepsilon) = 2\mathrm{i}\,\mathrm{Im}F_\pi(q^2)$. An isolated single intermediate state gives rise to a pole.

On general grounds, the analyticity of the scattering amplitudes (or form factors) was first derived by Gell-Mann, Goldberger, and Thirring from the condition of macroscopic causality, which states that commutators of field operators vanish when the points at which the operators are evaluated are separated by a spacelike interval. Then using the Cauchy theorem, we can write dispersion relations relating them to their imaginary parts. The once-subtracted dispersion relation for $F_\pi(q^2)$ is

$$F_\pi(q^2) = F_\pi(0) + \frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}F_\pi(s)}{s(s - q^2 - \mathrm{i}\epsilon)}\mathrm{d}s \ . \tag{10.17}$$

Of course the unsubtracted dispersion relation is equally valid, provided that the function $F_\pi(z)$ decreases rapidly [$F_\pi(z) \to 0$ as $|z| \to \infty$] to allow the integral to converge. In this case the imaginary part obeys the sum rule

$$F_\pi(q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}F_\pi(s)}{s - q^2 - \mathrm{i}\epsilon}\mathrm{d}s \ , \quad F_\pi(0) = 1 = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}F_\pi(s)}{s}\mathrm{d}s \ . \tag{10.18}$$

The computation of the form factor then reduces to evaluating its imaginary part. $\mathrm{Im}[F_\pi(s)]$ can be obtained from $\mathrm{e}^+ + \mathrm{e}^- \to \gamma^* \to \pi^+ + \pi^-$ data for which the propagator of the $\rho(770)$ meson dominates in the $s \sim 0.6\ \mathrm{GeV}^2$ region (see Fig. 10.2b). In the zero-width approximation of this ρ meson, we have from its propagator [note that $\mathrm{Im}(x \pm \mathrm{i}\epsilon)^{-1} = \mp\pi\delta(x)$],

$$\mathrm{Im}F_\pi(s) = \pi g_{\rho\pi\pi} f_\rho m_\rho \delta(s - m_\rho^2) + \cdots \ , \tag{10.19}$$

where the residue at the ρ pole is $g_{\rho\pi\pi} f_\rho m_\rho$ as shown by Fig. 10.2b. The dots denote other contributions beyond the ρ^0. The two parameters $g_{\rho\pi\pi}$ and

f_ρ in (19) are defined as follows: The dimensionless coupling constant $g_{\rho\pi\pi}$ can be determined from the $\rho(k) \to \pi(p) + \pi(p')$ strong decay for which the effective Lagrangian $\mathcal{L}_{\text{eff}}$ and the decay amplitude may be written as

$$\mathcal{L}_{\text{eff}} = g_{\rho\pi\pi}\rho^\mu(x)\left[\pi(x)\overset{\leftrightarrow}{\partial_\mu}\pi(x)\right] \;,\;\; \mathcal{M}(\rho\to\pi\pi) = g_{\rho\pi\pi}\varepsilon^\mu(k)(p-p')_\mu \;.$$

This gives : $$\Gamma_\rho \equiv \Gamma(\rho\to\pi\pi) = \frac{m_\rho g^2_{\rho\pi\pi}}{48\pi}\left(1-\frac{4m_\pi^2}{m_\rho^2}\right)^{3/2} \;. \tag{10.20}$$

The electromagnetic decay constant of the ρ^0, called f_ρ, is defined similarly to the weak decay constant of the pion $f_\pi \approx 131\,\text{MeV}$. This f_π, which governs the weak decay $\pi^+ \to \mathrm{W}^+ \to \mathrm{e}^+ + \nu$, is defined by $\langle 0\,|\,\mathrm{A}^\mu\,|\,\pi^+(k)\rangle = \mathrm{i}f_\pi k^\mu$ [Fig. 10.3a and (12.55), (13.32)]. The decay constant f_ρ determines the amplitude $\mathcal{M}(\rho^0 \to \gamma^* \to \mathrm{e}^+ + \mathrm{e}^-)$ (Fig. 10.3b):

$$\langle 0\,|\,J^\mu_{\text{em}}\,|\,\rho(k)\rangle = f_\rho m_\rho \varepsilon^\mu \;,\quad \left\langle \mathrm{e}^+(p'),\mathrm{e}^-(p)\,\middle|\,J^{\text{em}}_\mu\,\middle|\,0\right\rangle = \overline{u}(p)\gamma_\mu v(p') \;,$$
$$\mathcal{M}(\rho^0\to\mathrm{e}^+ + \mathrm{e}^-) = \mathrm{i}\,(-\mathrm{i}e)^2\left\langle \mathrm{e}^+(p'),\mathrm{e}^-(p)\,\middle|\,J^{\text{em}}_\mu\,\middle|\,0\right\rangle \frac{-\mathrm{i}}{k^2}\,\langle 0\,|\,J^\mu_{\text{em}}\,|\,\rho(k)\rangle \,,$$

from which : $$\Gamma(\rho^0\to\mathrm{e}^+ + \mathrm{e}^-) = \frac{4\pi\alpha^2 f_\rho^2}{3m_\rho} \;. \tag{10.21}$$

The decay constant f_ρ has the dimension of mass, as does f_π.

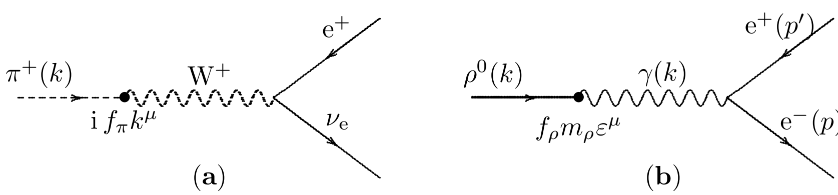

Fig. 10.3. (**a**) $\pi^+ \to \mathrm{e}^+ + \nu_\mathrm{e}$ weak decay; (**b**) $\rho^0 \to \mathrm{e}^+ + \mathrm{e}^-$ electromagnetic decay

Putting (19) into (17), we get

$$F_\pi(q^2) = 1 + \frac{g_{\rho\pi\pi}f_\rho}{m_\rho}\frac{q^2}{m_\rho^2 - q^2} + \cdots \;. \tag{10.22}$$

It turns out from (20) and (21) that data give $g^2_{\rho\pi\pi}/4\pi \approx 2.88$, $f_\rho \approx 150$ MeV. From these numbers, we get $g_{\rho\pi\pi}f_\rho/m_\rho \approx 1.17$, close to 1, which is the value corresponding to the universal ρ dominance hypothesis, according to which the ρ meson completely determines the electromagnetic form factors of low-lying hadrons (pion, nucleon). In the dispersion relation approach, this hypothesis consists in neglecting all contributions from non-resonant background ($\gamma^* \to$ continuum background $\to \pi^+ + \pi^-$), as well as from other resonances $f_0(1300)$, $\rho(1450)$, etc. This value 1 obviously satisfies the sum rule (18). Then, in the zero-width approximation of the ρ meson:

$$F_\pi(q^2) = \frac{1}{1-\frac{q^2}{m_\rho^2}} \longrightarrow \quad \text{pion radius } \sqrt{\langle r_\pi^2\rangle} = \frac{\sqrt{6}}{m_\rho} \approx 0.64\,\text{fm}\,. \tag{10.23}$$

A more sophisticated expression of $F_\pi(q^2)$ with a Breit–Wigner form and a q^2-dependent of the ρ-width, is given in (13.40) and (13.41).

We turn now to the weak form factors of $\overline{\text{B}} \to \text{D}$ transition. Instead of $f_\pm(q^2)$ in (15), we introduce another parameterization that singles out the spin character of the current,

$$\begin{aligned}\langle \text{D}(p')\,|\,\text{V}^\mu\,|\,\overline{\text{B}}(p)\rangle = G_1(q^2)&\left[(p+p')^\mu - \frac{M_B^2-M_D^2}{q^2}q^\mu\right]\\ &+ G_0(q^2)\frac{M_B^2-M_D^2}{q^2}q^\mu\,.\end{aligned} \tag{10.24}$$

Indeed let $q_\mu = (p-p')_\mu$ act on both the left and right members of the above equation, then we realize that $G_0(q^2)$ represents the matrix element of the operator $q_\mu \text{V}^\mu$ which behaves like a $J^P = 0^+$ scalar object. For a conserved current, this spin-0 form factor vanishes. The $G_1(q^2)$ form factor corresponds to the spin-1 part of the current since its associated operator is orthogonal to q_μ, i.e. $q_\mu\left[(p+p')^\mu - (M_B^2-M_D^2)q^\mu/q^2\right] = 0$. $G_1(q^2)$ and $G_0(q^2)$ are subject to $G_1(0) = G_0(0)$, which eliminates the spurious pole at $q^2=0$.

One advantage of considering $G_1(q^2)$ and $G_0(q^2)$ lies in the fact that their q^2 dependences are easy to guess, since the imaginary parts of these form factors are associated respectively with the vector B_c^* and scalar B_c resonances bearing the bottom–charm quantum numbers. To be more explicit, let us mimic the $F_\pi(q^2)$ case with the ρ dominance, then the $G_1(q^2)$ and $G_0(q^2)$ form factors as visualized by Fig. 10.4 are respectively dominated by the vector B_c^* and scalar $(J^P = 0^+, \text{B}_c)$ bottom–charm $\overline{\text{c}}\text{b}$ resonances of masses around 6.5 GeV, for which the q^2 dependences are monopoles, as in (23):

$$G_1(q^2) = \frac{G_1(0)}{1-q^2/M^2_{B_c^*}}\,, \qquad G_0(q^2) = \frac{G_0(0)}{1-q^2/M^2_{B_c}}\,. \tag{10.25}$$

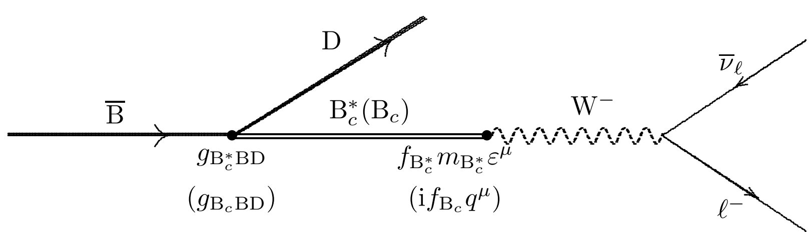

Fig. 10.4. $\text{B}_c^*\,(\text{B}_c)$ pole dominance of the $G_1(q^2)\,(G_0(q^2))$ form factors in weak decay $\overline{\text{B}} \to \text{D} + \ell^- + \overline{\nu}_\ell$

It must be emphasized that these q^2 behaviors, although plausible, are model dependent, they are derived from the assumption of the nearest resonance dominance. It could be a good approximation in the largest-q^2 region near the poles. Assuming this monopole q^2 dependence (25), and using (16), the weak form factors $G_{1,0}(q^2)$ are theoretically determined, including their normalizations $G_{1,0}(0)$. B meson decays will be discussed in detail in Chap. 16; as shown by (16.86) the q^2 dependence of $G_1(q^2) = f_+(q^2)$ can be experimentally measured by looking at the q^2 distribution of $\mathrm{d}\Gamma(\mathrm{B} \to \overline{\mathrm{D}} + e^+ + \nu_e)/\mathrm{d}q^2$, therefore the $G_1(q^2)$ extracted from data can be confronted with theoretical models of form factor. As for the scalar form factor $G_0(q^2) = f_+(q^2) + \left[q^2/(M_\mathrm{B}^2 - M_\mathrm{D}^2)\right] f_-(q^2)$, its contribution to $\Gamma(\mathrm{B} \to \overline{\mathrm{D}} + \ell^+ + \nu_\ell)$ via $f_-(q^2)$ is proportional to the squared lepton mass [see (16.87)], hence only the τ in the decay products is sensitive to $G_0(q^2)$.

This section ends with a discussion of the Watson theorem according to which, below the inelastic threshold, the phases of electromagnetic (weak) amplitudes are equal to the strong interaction elastic phase-shifts of the hadrons involved. The demonstration is based on the unitarity of the S matrix together with the time-reversal invariance of the amplitudes. Since $S^\dagger S = 1$, $S = 1 - \mathrm{i}\,\mathcal{T}$, then

$$\mathrm{i}\big(\mathcal{T}_{\mathrm{f\,i}} - \mathcal{T}^*_{\mathrm{i\,f}}\big) = \sum_n \mathcal{T}^*_{n\,\mathrm{f}}\mathcal{T}_{n\,\mathrm{i}}\ ,\ \mathcal{T}_{\mathrm{f\,i}} \equiv \langle \mathrm{f}\,|\,\mathcal{T}\,|\,\mathrm{i}\rangle\ .$$

Consider the electromagnetic transition $A \to B + C$ followed by an elastic final-state strong interaction: $A \xrightarrow{\text{em.}} B + C \xrightarrow{\text{strong}} B + C$, i.e. we have i$=A$ and n=f=$B + C$. Time-reversal invariance implies $\mathcal{T}_{f\,i} = \mathcal{T}_{\overline{i}\,\overline{f}}$ (where $\left|\overline{i}\right\rangle = \mathcal{T}\,|i\rangle$), such that the left-hand side of the above unitarity relation reduces to $-2\mathrm{Im}\,\mathcal{T}_{f\,i}$ and both sides are real valued. Denoting the electromagnetic amplitude by $|T_\mathrm{em}|\exp(\mathrm{i}\,\varphi)$, and the strong amplitude by $|T_\mathrm{sg}|\exp(\mathrm{i}\,\delta)$, then $\varphi = \delta$. For instance, consider the electromagnetic form factor $\gamma^* \xrightarrow{\text{em.}} \pi + \pi$ followed by a final-state strong interaction between the pions $\pi+\pi \xrightarrow{\text{strong}} \pi+\pi$, then $F_\pi(q^2)$ is given by $|F_\pi(q^2)|\exp[\mathrm{i}\delta_{\pi\pi}(q^2)]$. Since the strong phase-shifts can in principle be extracted from experimental data, in particular from $\pi + \pi$ scattering, the form factor $F_\pi(q^2)$ can be obtained from the p-wave phase-shift $\delta_{\pi\pi}(q^2)$ via dispersion relation (Problem 10.7). The same theorem applies to the weak form factor $G_1(q^2)$ [or $G_0(q^2)$], its phase is the p-wave (or s-wave) phase-shift of the B+D $\to$ B+D strong scattering.

10.3 Exclusive Reaction: Elastic Scattering

According to the Feynman rules, the one-photon exchange elastic amplitude $\mathrm{e}^-(p) + \mathrm{N}(P) \to \mathrm{e}^-(p') + \mathrm{N}(P')$ (Figs. 4.9 and 10.6a) may be written as

$$\mathcal{M} = (+\mathrm{i}e)\overline{u}(p')\gamma^\mu u(p)\left(\frac{-\mathrm{i}}{q^2}\right)(-\mathrm{i}e)H_\mu\,,$$

$$H_\mu \equiv \overline{u}(P')\left[\gamma_\mu F_1(q^2) + \mathrm{i}\frac{\sigma_{\mu\nu}q^\nu}{2M}F_2(q^2)\right]u(P)\ , \tag{10.26}$$

where $q = P' - P$. It can be deduced directly from the $\mathrm{e}^- + \mu^+ \to \mathrm{e}^- + \mu^+$ amplitude by a simple substitution $\gamma_\mu \Rightarrow \gamma_\mu F_1(q^2) + \mathrm{i}[\sigma_{\mu\nu} q^\nu/2M] F_2(q^2)$ at the muon vertex.

We put $K = P + P'$, $s = (P + p)^2$, $-t = -q^2 = Q^2 \geq 0$. The nucleon and electron masses are respectively denoted by M and m. The cross-section is given by formulas (4.57) and (4.62):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{1}{16\pi\lambda(s, M^2, m^2)} \Big(\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \Big) . \tag{10.27}$$

The factor $1/4 = (1/2)(1/2)$ in (27) represents the spin averaging of the incoming electron and proton; for undetected spins in the final states, their summation is understood. With the Gordon decomposition, let us rewrite the nucleon current H_μ as $\overline{u}(P')\,[(F_1 + F_2)\gamma_\mu - (K_\mu/2M)F_2]\, u(P)$, then

$$\begin{aligned}
\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \left(\frac{e^2}{q^2}\right)^2 l^{\mu\nu} H_{\mu\nu} \ , \\
H_{\mu\nu} &= \frac{1}{2} \Big\{ (F_1 + F_2)^2 \mathrm{Tr}\, [\not{P}'\gamma_\mu \not{P}\gamma_\nu + M^2 \gamma_\mu\gamma_\nu] \\
&\quad + F_2^2 \frac{K_\mu K_\nu}{4M^2} \mathrm{Tr}\, [\not{P}' \not{P} + M^2] - 4F_2(F_1 + F_2) K_\mu K_\nu \Big\} \ , \\
l^{\mu\nu} &= \frac{1}{2} \mathrm{Tr}\, [\not{p}'\gamma^\mu \not{p}\gamma^\nu + m^2 \gamma^\mu\gamma^\nu] = 2\left(p^\mu p'^\nu + p^\nu p'^\mu + \frac{q^2 g^{\mu\nu}}{2} \right) \ ,
\end{aligned} \tag{10.28}$$

so that

$$\begin{aligned}
\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= 4\left(\frac{e^2}{q^2}\right)^2 \Bigg\{ (F_1 + F_2)^2 \left[(s - M^2 - m^2)^2 + q^2 (s + \frac{q^2}{2}) \right] \\
&\qquad - \left[2F_1F_2 + \left(1 + \frac{q^2}{4M^2}\right) F_2^2 \right] \left[(s - M^2 - m^2)^2 + q^2(s - m^2) \right] \Bigg\} \\
&= 4\left(\frac{e^2}{q^2}\right)^2 \Bigg\{ \left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) [(s - M^2 - m^2)^2 + q^2(s - m^2)] \\
&\qquad + (F_1 + F_2)^2 q^2 \left(m^2 + \frac{q^2}{2} \right) \Bigg\} \ .
\end{aligned} \tag{10.29}$$

Following Sachs, let us define the electric $G_{\mathrm{E}}(q^2)$ and magnetic $G_{\mathrm{M}}(q^2)$ form factors as a combination of $F_1(q^2)$ and $F_2(q^2)$:

$$\begin{aligned}
& G_{\mathrm{E}}(q^2) = F_1(q^2) + \frac{q^2}{4M^2} F_2(q^2) \quad , \quad G_{\mathrm{M}}(q^2) = F_1(q^2) + F_2(q^2) \ , \\
& G_E^{\mathrm{p}}(0) = 1 \ \ , \ \ G_E^{\mathrm{n}}(0) = 0 \ \ , \ \ G_M^{\mathrm{p}}(0) = 2.79 \ \ , \ \ G_M^{\mathrm{n}}(0) = -1.91 \ , \\
& F_1^2 - \frac{q^2}{4M^2} F_2^2 = \frac{G_E^2 - (q^2/4M^2) G_M^2}{1 - (q^2/4M^2)} \ .
\end{aligned} \tag{10.30}$$

In the *laboratory* system $P = (M, \mathbf{0})$, $p = (E, \boldsymbol{p})$, $p' = (E', \boldsymbol{p}')$. Using $P'^2 = M^2 = (P+q)^2$, we deduce

$$q^2 + 2M(E - E') = 0 \quad , \quad q^2 = -2M\nu \; , \; \nu \equiv E - E' \; . \tag{10.31}$$

In the limit $s \gg m^2$, the incoming and outgoing electrons are extremely relativistic, the formula (4.69) for the cross-section is adapted to this case. We also have

$$\boldsymbol{p} \cdot \boldsymbol{p}' \approx EE' \cos\theta \quad ; \quad q^2 \approx -2EE'(1-\cos\theta) = -4EE' \sin^2\frac{\theta}{2} \; . \tag{10.32}$$

From (31) and (32): $q^2 = -2M(E-E') = -2EE'(1-\cos\theta)$, we remark that for fixed E the two quantities E' and $\cos\theta$ are not independent because

$$\frac{E'}{E} = \frac{1}{[1 + \frac{E}{M}(1-\cos\theta)]} = \frac{1}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}} \; . \tag{10.33}$$

Coming back to the first term of the right-hand side of (29), we find $(s - M^2 - m^2)^2 + q^2(s - m^2) \approx 4M^2 EE' \cos^2\frac{\theta}{2}$. When we rewrite the $q^2(m^2 + q^2/2) \approx q^2(q^2/2)$ of the second term as $(-q^2/2M^2)\tan^2\frac{\theta}{2}$ times the common factor $4M^2EE'\cos^2\frac{\theta}{2}$, we obtain the following results, using (4.64), (4.69) and the relation $\mathrm{d}\Omega_{\mathrm{lab}} = \pi\,\mathrm{d}Q^2/E'^2$, $(e^2/4\pi = \alpha \approx 1/137)$:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} &= \frac{\pi\,\sigma_{\mathrm{Mott}}}{EE'}\left[\frac{G_{\mathrm{E}}^2(q^2) - \frac{q^2}{4M^2}G_{\mathrm{M}}^2(q^2)}{1 - \frac{q^2}{4M^2}} - \frac{q^2}{2M^2}G_{\mathrm{M}}^2(q^2)\tan^2\frac{\theta}{2}\right] \; , \\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}} &= \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}}\left[\frac{G_{\mathrm{E}}^2(q^2) - \frac{q^2}{4M^2}G_{\mathrm{M}}^2(q^2)}{1 - \frac{q^2}{4M^2}} - \frac{q^2}{2M^2}G_{\mathrm{M}}^2(q^2)\tan^2\frac{\theta}{2}\right] \; , \\
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}} &\equiv \frac{\sigma_{\mathrm{Mott}}}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}} \; , \text{ and } \sigma_{\mathrm{Mott}} \equiv \left(\frac{\alpha\;\cos\frac{\theta}{2}}{2E\sin^2\frac{\theta}{2}}\right)^2 \; .
\end{aligned} \tag{10.34}$$

Remarks. From the Rosenbluth formula (34), three remarks can be made: (a) For a *structureless* nucleon: $F_1^{\mathrm{p}}(q^2) = 1$, $F_1^{\mathrm{n}}(q^2) = F_2^{\mathrm{p}}(q^2) = F_2^{\mathrm{n}}(q^2) = 0$; $G_{\mathrm{E}}^{\mathrm{p}}(q^2) = G_{\mathrm{M}}^{\mathrm{p}}(q^2) = 1$, $G_{\mathrm{E}}^{\mathrm{n}}(q^2) = G_{\mathrm{M}}^{\mathrm{n}}(q^2) = 0$. The elastic cross-section $\mathrm{e}^- +$ *pointlike* neutron is identically vanishing. We also recover the formula (4.161) of a *pointlike* proton. This formula (4.161) or (34) also gives the elastic $\mathrm{e}^- + \mu^+$ cross-section

$$\begin{aligned}
\frac{\mathrm{d}\sigma(\mathrm{e}^- + \mu^+ \to \mathrm{e}^- + \mu^+)}{\mathrm{d}Q^2} &= \frac{\pi\sigma_{\mathrm{Mott}}}{EE'}\left(1 - \frac{q^2}{2M^2}\tan^2\frac{\theta}{2}\right) \; , \\
\frac{\mathrm{d}\sigma(\mathrm{e}^- + \mu^+ \to \mathrm{e}^- + \mu^+)}{\mathrm{d}\Omega_{\mathrm{lab}}} &= \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}}\right)_{NS}\left(1 - \frac{q^2}{2M^2}\tan^2\frac{\theta}{2}\right) \; .
\end{aligned} \tag{10.35}$$

The subscript NS in $(\mathrm{d}\sigma/\mathrm{d}\Omega_{\mathrm{lab}})_{\mathrm{NS}}$ denotes *no-structure* cross-section. The first term $\sigma_{\mathrm{Mott}} \equiv \left[\alpha\ \cos\frac{\theta}{2}/2E\sin^2\frac{\theta}{2}\right]^2$ in (34) [already met in (4.159)] corresponds to the scattering of a spin-$\frac{1}{2}$ pointlike charged particle by a spinless pointlike target of charge $\pm e$. The second term $1/(1+\frac{2E}{M}\sin^2\frac{\theta}{2})$ represents the target recoil where the generic M denotes the target mass. Without the $\cos^2\frac{\theta}{2}$ term, the Mott cross-section is identical to the Rutherford cross-section $\sigma_{\mathrm{R}} \equiv [\alpha/2E\sin^2\frac{\theta}{2}]^2$ which corresponds to the scattering of a scalar projectile by an ultra heavy scalar target, both are pointlike with charges $\pm e$. The angular distribution $\cos^2\frac{\theta}{2}$ reflects the spin-$\frac{1}{2}$ of the *projectile*.
(b) The term $\tan^2\frac{\theta}{2}$ of (34), (35) comes from $G_{\mathrm{M}}^2 \equiv (F_1+F_2)^2$ presented on the last line of the RHS of (29). Through the magnetic moment, it represents the target spin. We recall that a Dirac particle, even pointlike, has a spin magnetic moment approximately equal to a Bohr magneton. The $\tan^2\frac{\theta}{2}$ corresponds to the spin-$\frac{1}{2}$ effect of the *target*: if the latter is spinless, as in electron scattering by pion, this term is absent, and the Mott cross-section σ_{Mott} is recovered. Note the coefficient $q^2/2M^2$ of the magnetic term which becomes important for large q^2. In brief, the $\cos^2\frac{\theta}{2}$ refers to the spin-$\frac{1}{2}$ of the projectile, the $\tan^2\frac{\theta}{2}$ carries the spin-$\frac{1}{2}$ character of the target. The electron–pion scattering therefore can be immediately deduced:

$$\frac{\mathrm{d}\sigma(\mathrm{e}^- + \pi^\pm \to \mathrm{e}^- + \pi^\pm)}{\mathrm{d}\Omega_{\mathrm{lab}}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}} |F_\pi(q^2)|^2\ . \qquad (10.36)$$

Similarly, we get the cross-section of electron scattering by a spinless nucleus $\mathcal{N}$ of charge Ze considered as an example of form factors in Sect. 10.1. Thus,

$$\frac{\mathrm{d}\sigma(\mathrm{e}^- + \mathcal{N} \to \mathrm{e}^- + \mathcal{N})}{\mathrm{d}\Omega_{\mathrm{lab}}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{lab}}}\right)_{\mathrm{NS}} |F_{\mathrm{N}}(q^2)/e|^2\ , \qquad (10.37)$$

which tends to $Z^2\,(\mathrm{d}\sigma/\mathrm{d}\Omega_{\mathrm{lab}})_{\mathrm{NS}}$ for a pointlike spinless nucleus.

We will see that in deep inelastic lepton–nucleon scattering, quarks are revealed as constituents of the nucleon through the $\tan^2\frac{\theta}{2}$ term. The physical meaning of the $\cos^2\frac{\theta}{2}$ and $\tan^2\frac{\theta}{2}$ terms (or $\sin^2\frac{\theta}{2}$ for the latter if we do not factorize the common $\cos^2\frac{\theta}{2}$) can be understood as follows.

The electromagnetic interactions of the projectile electron may be split into two parts. The first, composed of $\overline{u}_{\mathrm{L}}\gamma^\mu u_{\mathrm{L}}$ and $\overline{u}_{\mathrm{R}}\gamma^\mu u_{\mathrm{R}}$, corresponds to its electric interaction with another charged target for which the outgoing and incoming electrons conserve their helicities. The second, $\overline{u}_{\mathrm{L}}[i\sigma^{\mu\nu}q_\nu]u_{\mathrm{R}}$ and $\overline{u}_{\mathrm{R}}[i\sigma^{\mu\nu}q_\nu]u_{\mathrm{L}}$, corresponds to its magnetic interaction with the target spin for which the outgoing and incoming electron have opposite helicities. Angular momentum conservation implies that the forward (backward) scattering is allowed (forbidden) by the electric interaction and forbidden (allowed) by the magnetic interaction. The helicity-conserving amplitude is proportional to the rotation matrix $d^{1/2}_{1/2,1/2}(\theta) = d^{1/2}_{-1/2,-1/2}(\theta) = \cos\frac{\theta}{2}$, characteristic of a spin 1/2 particle (the electron) scattered by any charged (spinless or not) target; while the helicity reversing amplitude depends on $d^{1/2}_{1/2,-1/2}(\theta) = -d^{1/2}_{-1/2,1/2}(\theta) = \sin\frac{\theta}{2}$, which can only occur with an electron scattered by a spin-1/2 target having a magnetic moment [remember $\sigma_{\mu\nu} \sim \boldsymbol{\sigma}$ with (10)]. The diagrams in Fig. 10.5 illustrate the meaning of the $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$ angular distributions.

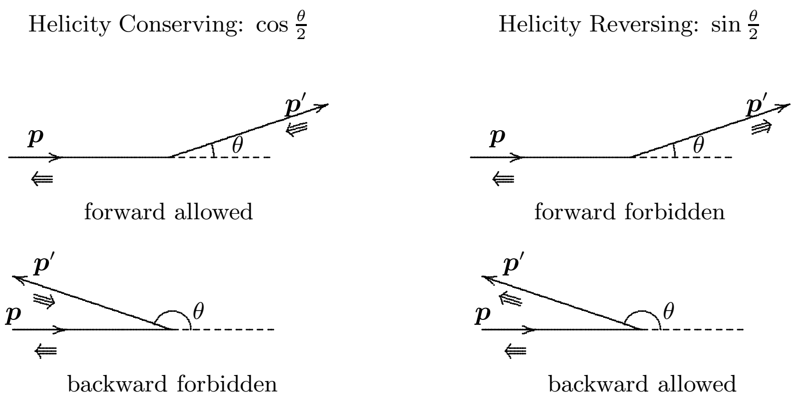

Fig. 10.5. Angular distributions from helicity arguments

(c) The angular distribution $[\sin\frac{\theta}{2}]^{-4}$ is intimately connected to the discovery of nuclei. Until today, from atoms to nuclei, from nuclei to nucleons, from nucleons to quarks, the successive layers of matter are revealed in striking similarity with the Rutherford discovery of nuclei. Before Rutherford, atoms were considered to be the most fundamental objects. Since an atom is electrically neutral and contains light particles of negative charges (electrons), the question was how its very heavy (compared to the electron) and positive charged particles were distributed. If the positive charges were uniformly spread over the whole atomic volume as was generally thought at the time (the model of Thomson, the electron's discoverer), then by projecting energetic He ions (α particles) on thin metal foils, Geiger and Marsden expected that the α beam would be deflected by a small angle. To their surprise, they observed that a non-negligible number of projectiles were bounced back by an angle $\theta \geq 90^0$. It took Rutherford two years to find the explanation.

If the Thomson model of charge distribution were correct, such large deflections could only arise, according to Rutherford, from a multitude of scatterings by a huge number of atoms of matter. However a simple calculation based on the theory of probability shows that the odds for finding an event with $\theta \geq 90^0$ are vanishingly small, something like 10^{-3500} and not 1/20 000 as found by Geiger and Marsden. The simple power law $[\sin\frac{\theta}{2}]^{-4}$ computed by Rutherford and experimentally verified by his group can only be explained if the projectile strikes a hard obstacle, its charge must be concentrated in an extremely small region inside the atom, it cannot be spread over the large atomic volume. The physical meaning of the angular distribution $[\sin\frac{\theta}{2}]^{-4}$ is clear. This term proportional to $(1/q^2)^2$ comes from the propagator of a photon probing a pointlike charged target found at the center of a much larger atom. If this were not the case, i.e. if the charges were distributed at random over the whole atomic volume and hit by the photon, then the atomic form factor, which decreases very rapidly for large q^2 (large θ) because of its size, would make vanishingly small the probability for observing an event with large θ. Thus was born the notion of a pointlike nucleus of the atom, in which the atom contains a hard constituent: a small, massive and positively charged nucleus. After nearly one hundred years, this remarkable feature is still relevant today. In all scattering processes, a large deflection of the projectile (or large transverse momentum P_T in our contemporary language) is synonymous with a hard constituent in the target, whether the latter is an atom or a hadron. Events in which a projectile is bounced back by $\approx 180^0$ are really spectacular and may reveal new things.■

After these remarks, we come back to the Rosenbluth formula (34). For each fixed q^2, if we plot experimental data of $\mathrm{d}\sigma/\mathrm{d}\Omega_{\mathrm{lab}}$ as a function of $\tan^2\frac{\theta}{2}$, its linear form $A(q^2)+B(q^2)\tan^2\frac{\theta}{2}$ allows us to separate $A(q^2)$ and $B(q^2)$ and to extract the form factors $G_{\mathrm{E}}(q^2)$ and $G_{\mathrm{M}}(q^2)$. From studies of the elastic scattering $\mathrm{e}+\mathrm{N}\to\mathrm{e}+\mathrm{N}$, $G_{\mathrm{E}}(q^2)$ and $G_{\mathrm{M}}(q^2)$ are found to decrease as a *dipole mode*,

$$\frac{G_E^{\mathrm{p}}(q^2)}{G_E^{\mathrm{p}}(0)}=\frac{G_M^{\mathrm{p}}(q^2)}{G_M^{\mathrm{p}}(0)}=\frac{G_M^{\mathrm{n}}(q^2)}{G_M^{\mathrm{n}}(0)}=\frac{1}{\left(1-\frac{q^2}{\Lambda^2}\right)^2}\quad;\quad G_E^{\mathrm{n}}(q^2)=0\;,\tag{10.38}$$

with $\Lambda=0.84\,\mathrm{GeV}$. By doing inverse Fourier transform of the nonrelativistic limit of $[1-(q^2/\Lambda^2)]^{-2}$, we obtain $\rho(r=|\boldsymbol{x}|)$, the electric charge distribution in the proton, together with its squared radius $\langle r_{\mathrm{p}}^2\rangle$:

$$\begin{aligned}\rho(r)&=\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{(1+|\boldsymbol{q}|^2/\Lambda^2)^2}=\frac{\Lambda^3}{8\pi}\mathrm{e}^{-\Lambda r}\quad,\quad\int\mathrm{d}^3x\rho(r)=1\;,\\ \sqrt{\langle r_{\mathrm{p}}^2\rangle}&=\frac{\sqrt{12}}{\Lambda}\approx 0.8\ \mathrm{fm}\;.\end{aligned}\tag{10.39}$$

The above distribution $\rho(r)=\Lambda^3\mathrm{e}^{-\Lambda r}/(8\pi)$ has the property that $\rho(r)\to$ constant as $r\to 0$, indicating that there is no hard core in the nucleon, it is not like a plum with a stone in the middle. The hard core corresponds for example to a monopole decreasing form factor that gives a spatial distribution $\rho(r)=\Lambda^2\mathrm{e}^{-\Lambda r}/(4\pi r)$ tending to infinity as $r\to 0$ (Problem 10.6). The question is: Is the proton like jelly or like a pomegranate? Intuitively we may imagine the former configuration as an interpretation of the bootstrap nuclear democracy concept, for which hadrons are composite of themselves and no one is more elementary than the others. For example the proton may be a bound state of $\mathrm{n}+\pi^+$ or $\Lambda+\mathrm{K}^+$ or $\Sigma^++\mathrm{K}^0$. As we will see, deep inelastic scattering provides a definite answer in favor of the second interpretation according to which the proton contains pointlike constituents. As for the neutron, it is *a priori* not at all guaranteed that the electromagnetic e–n cross-section turns out to be also nonzero, as the neutron is electrically neutral. Since this is the case, the constituents of the latter must be charged and must be the sources of the neutron anomalous magnetic moment $G_M^{\mathrm{n}}(0) = -1.91\mu_B$. Finally, to connect the electromagnetic form factors to the weak form factors via the conserved vector current (CVC) property of the $\Delta S = 0$ weak vector current (Chap. 12), it is useful to write down the isovector and isoscalar form factors previously defined in (13) and (30), using experimental data on $G_{\mathrm{E}}(q^2)$ and $G_{\mathrm{M}}(q^2)$ in (38). We have

$$F_1^1(q^2) = \frac{1-4.70\,u(q^2)}{(1-u(q^2))[1-v(q^2)]^2}\ ;\ F_1^0(q^2) = \frac{1-0.88\,u(q^2)}{(1-u(q^2))[1-v(q^2)]^2}\ ;$$
$$F_2^1(q^2) = \frac{3.70}{(1-u(q^2))[1-v(q^2)]^2}\ ;\ F_2^0(q^2) = \frac{-0.12}{(1-u(q^2))[1-v(q^2)]^2}$$
$$\text{where}\quad u(q^2) \equiv \frac{q^2}{4M^2}\ ,\ v(q^2) \equiv \frac{q^2}{\Lambda^2}\ . \tag{10.40}$$

By CVC, the isovector form factors $F_1^1(q^2)$ and $F_2^1(q^2)$ are the same form factors of the vectorial V part of the V − A weak charged current involved in nucleon β-decay and in neutrino–nucleon elastic scattering (Chap. 12).

To close the section, let us mention that the experimental range of the momentum transfer q^2 reached today at the electron–proton (positron–proton) HERA collider in Hamburg is $Q^2 \geq 2000$ GeV^2 for which the form factors $G_{\mathrm{E,M}}(q^2)$ squared, as given by (38), decreases from 1 to 10^{-14} or less, in sharp contrast with the nearly constant behavior of the structure functions that we are going to discuss now. The reason is that the nucleon contains pointlike constituents, as we will see.

10.4 Inclusive Reaction: Deep Inelastic Scattering

An exclusive reaction is a process in which the final state contains a limited number of particles effectively observed. Elastic or quasielastic lepton scattering from nucleon $\mathrm{e}+\mathrm{N}\to\mathrm{e}+\mathrm{N}$, $\mathrm{e}+\mathrm{N}\to\mathrm{e}+\mathrm{N}^*$, $\nu_\mu+\mathrm{N}\to\mu^-+\mathrm{N}+\pi$ are some examples.

On the other hand, in an inclusive reaction $\mathrm{e}+\mathrm{N}\to\mathrm{e}+X$ (more generally $\ell+\mathrm{N}\to\ell'+X$) only the final lepton ℓ' is detected, no attempt is made to select a particular hadronic channel. These unobserved hadrons are symbolically designated by X. Since each exclusive cross-section decreases sharply as the squared of its corresponding form factors, the inclusive one, which is nothing but the sum of all possible exclusive modes, would at first sight be negligible for large q^2. When the inclusive reaction $\mathrm{e}+\mathrm{N}\to\mathrm{e}+X$ was getting ready to be measured at SLAC (Stanford) in the 1960s, the general impression was rather pessimistic about the number of events that could be collected. That was without allowing for the remarkable intuition of Bjorken who among the first showed that deep inelastic scattering (inclusive reaction at large q^2) was an ideal tool to probe the nucleon constituents. He also predicted the *scaling law* for the nucleon structure functions, later confirmed by experiments. This law tells us that, once the term $1/Q^4$ is subtracted, the deep inelastic cross-section would be constant [see (10.66) below], rather than rapidly decreasing as the squared form factors of an exclusive reaction. This $1/Q^4$ term (from the one-photon exchange) is common to both inclusive and exclusive cross-sections [see (29) and (34)].

This surprising discovery is at the origin of the *parton* model, the name was proposed by Feynman to denote the free and pointlike constituents of hadrons in his explanation of the Bjorken scaling law. At high energy, it is intuitively conceivable that deep inelastic reaction represents the sum of scatterings by partons. Since these are pointlike, there are no decreasing form factors and the cross-section behaves differently from the exclusive one.

10.4.1 Structure Functions

Before considering the details of the dynamics, let us first examine the kinematics of exclusive (elastic) and inclusive electron–nucleon reactions shown in Figs. 10.6a and 10.6b respectively. For a two-body → two-body scattering, for example the $\mathrm{e}(p)+\mathrm{N}(P) \to \mathrm{e}(p')+\mathrm{N}(P')$ or $\mathrm{e}(p)+\mathrm{N}(P) \to \mathrm{e}(p')+\mathrm{N}^*(P')$, the cross-section depends kinematically on two independent variables which can be chosen as the Mandelstam invariants $s \equiv (P+p)^2 = (P'+p')^2$ and $t \equiv q^2 = (p-p')^2 \equiv -Q^2$. In the laboratory system $P = (M, \mathbf{0})$, $p = (E, \boldsymbol{p})$, $p' = (E', \boldsymbol{p}')$, another kinematic variable $\nu = P.q/M = E - E'$ which represents the energy loss is frequently used. The two independent variables can be taken as E and ν, instead of s and t.

On the other hand, in an inclusive reaction $\mathrm{e}(p) + \mathrm{N}(P) \to \mathrm{e}(p') + X$, since no specific hadron is selected, the squared invariant mass of unobserved hadronic states X is free to take all continuous values $\geq M^2$. Unlike (31), now $p_X^2 \equiv (P+q)^2 = M^2 + q^2 + 2M\nu$ is no more constrained to be equal to the squared mass of any specific hadron observed in a two-body reaction, i.e. q^2 and ν are independent. The inclusive cross-section depends on three kinematic variables that may be taken as s , q^2, and ν. In the one-photon exchange, the elastic cross-section $\mathrm{d}\sigma_{\mathrm{el}}$: $\mathrm{e}(p) + \mathrm{N}(P) \to \mathrm{e}(p') + \mathrm{N}(P')$ and the inclusive one $\mathrm{d}\sigma_{\mathrm{in}}$: $\mathrm{e}(p) + \mathrm{N}(P) \to \mathrm{e}(p') + X$ are given by (Chap. 4)

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{el}} &= \frac{(2\pi)^4\delta^4(p'+P'-p-P)}{2\sqrt{\lambda(s,M^2,m^2)}} \left(\frac{e^2}{q^2}\right)^2 l^{\mu\nu} H_{\mu\nu} \frac{\mathrm{d}^3p'}{(2\pi)^3 2E_{p'}} \frac{\mathrm{d}^3P'}{(2\pi)^3 2E_{P'}} , \\ \mathrm{d}\sigma_{\mathrm{in}} &= \frac{1}{2\sqrt{\lambda(s,M^2,m^2)}} \left(\frac{e^2}{q^2}\right)^2 l^{\mu\nu} W_{\mu\nu} \frac{\mathrm{d}^3p'}{(2\pi)^3 2E_{p'}} , \end{aligned} \tag{10.41}$$

in which the leptonic and the nucleonic tensors $l^{\mu\nu}$, $H_{\mu\nu}$ are defined in (28).

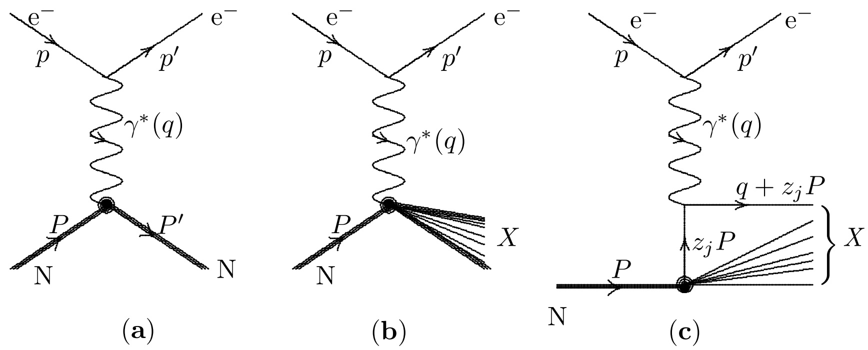

Fig. 10.6. (**a**) Elastic electron–nucleon (e–N) scattering; (**b**) deep inelastic e–N scattering; (**c**) electron–parton scattering

Using the relation $2s\ \mathrm{d}Q^2 = \lambda(s, M^2, m^2)\ \mathrm{d}\cos\theta$ and the Appendix formulas, the integration over the solid angle $\mathrm{d}\Omega = 2\pi\,\mathrm{d}\cos\theta$ in $\mathrm{d}\sigma_{\mathrm{el}}$ gives

$$\int \frac{\mathrm{d}^3 p'}{(2\pi)^3 2E_{p'}} \frac{\mathrm{d}^3 P'}{(2\pi)^3 2E_{P'}} (2\pi)^4 \delta^4(p' + P' - p - P) = \int \frac{\mathrm{d}Q^2}{8\pi\sqrt{\lambda(s, M^2, m^2)}} \ ,$$

and (27) is recovered. In (41) the sum over the unobserved hadronic states denoted as $W_{\mu\nu}(P, q)$ can be written similarly to $H_{\mu\nu}(P, q)$:

$$\begin{aligned} H_{\mu\nu}(P,q) &= \tfrac{1}{2} \left\langle \mathrm{N}(P) \,\middle|\, J^{\mathrm{em}}_\mu \,\middle|\, \mathrm{N}(P') \right\rangle \left\langle \mathrm{N}(P') \,\middle|\, J^{\mathrm{em}}_\nu \,\middle|\, \mathrm{N}(P) \right\rangle \ , \\ W_{\mu\nu}(P,q) &= \tfrac{1}{2} \sum_X \Big\{ \left\langle \mathrm{N}(P) \,\middle|\, J^{\mathrm{em}}_\mu \,\middle|\, X, p_X \right\rangle \left\langle X, p_X \,\middle|\, J^{\mathrm{em}}_\nu \,\middle|\, \mathrm{N}(P) \right\rangle \\ &\qquad \times (2\pi)^4 \delta^4(P + q - p_X) \Big\} \ . \end{aligned} \tag{10.42}$$

In (42) the sum over final spin states is understood, and the average of the initial nucleon spin is explicit with the factor $\frac{1}{2}$. If we count the dimensions of different terms in the left-hand and the right-hand sides of (41) [note that $\delta^4(K)$ has the $(\mathrm{mass})^{-4}$ dimension], we deduce that $H_{\mu\nu}(P, q)$ must have a $(\mathrm{mass})^2$ dimension which is confirmed by (28). On the other hand, $W_{\mu\nu}(P, q)$ must be dimensionless, in agreement with the limit $\sigma_{\mathrm{in}} \to \sigma_{\mathrm{el}}$ where the sum over the X states, represented by the symbol Σ_X in (42), is reduced to a simple nucleon $\mathrm{N}(P') : \Sigma_X \ \ \to \ \ \int \mathrm{d}^3 P'/[(2\pi)^3 2E_{P'}]$. The latter, which has the $(\mathrm{mass})^2$ dimension, makes $W_{\mu\nu}(P, q)$ dimensionless.

By analogy with $H_{\mu\nu}(P, q)$ in (28), the most general form of $W_{\mu\nu}(P, q)$ depends on $g_{\mu\nu}$ and three other tensors made up of P, q, and is symmetric in the interchange $\mu \leftrightarrow \nu$ since $l^{\mu\nu}$ is. They are $(P_\mu q_\nu + q_\mu P_\nu)$, $P_\mu P_\nu$, and $q_\mu q_\nu$. Moreover, from the conserved electromagnetic current, $W_{\mu\nu}(P, q)$ must satisfy the two conditions $q^\mu W_{\mu\nu}(P, q) = q^\nu W_{\mu\nu}(P, q) = 0$. The four terms $g_{\mu\nu}$, $P_\mu P_\nu$, $q_\mu q_\nu$, and $(P_\mu q_\nu + P_\nu q_\mu)$ are then reduced to two that can be chosen as the dimensionless tensors $T^1_{\mu\nu}$ and $T^2_{\mu\nu}$, which are separately conserved ($q^\mu T^1_{\mu\nu} = q^\nu T^1_{\mu\nu} = 0$, $q^\mu T^2_{\mu\nu} = q^\nu T^2_{\mu\nu} = 0$),

$$T^1_{\mu\nu} = -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \quad , \quad T^2_{\mu\nu} = \frac{1}{M^2}\left(P_\mu - \frac{P.q}{q^2}q_\mu\right)\left(P_\nu - \frac{P.q}{q^2}q_\nu\right) \ .$$

Conventionally, we define

$$W_{\mu\nu}(P,q) = 4\pi\left[T^1_{\mu\nu}W_1(q^2,\nu) + T^2_{\mu\nu}W_2(q^2,\nu)\right] \ . \tag{10.43}$$

$W_1(q^2,\nu)$ and $W_2(q^2,\nu)$ are called the nucleon *structure functions*. Like $W_{\mu\nu}$, they are dimensionless and depend on two variables usually taken as q^2 and ν. In (43), we make explicit the factor 4π coming from the angular integration of the one-particle state d^3p_X in (42). Putting (43) into (41), we find in the nucleon rest frame

$$l^{\mu\nu}T^1_{\mu\nu} = 8EE'\sin^2\frac{\theta}{2} \ , \ l^{\mu\nu}T^2_{\mu\nu} = 4EE'\cos^2\frac{\theta}{2} \ , \ \frac{\mathrm{d}^3p'}{2E_{p'}} = \frac{\pi}{2E}\mathrm{d}Q^2\mathrm{d}\nu \ .$$

Then (41) becomes

$$\frac{\mathrm{d}\sigma_{\mathrm{in}}(\mathrm{e}+\mathrm{N}\to\mathrm{e}+X)}{\mathrm{d}Q^2\mathrm{d}\nu} = \frac{\pi\,\sigma_{\mathrm{Mott}}}{MEE'}\left[W_2(q^2,\nu) + 2\,W_1(q^2,\nu)\tan^2\frac{\theta}{2}\right] \ . \tag{10.44}$$

This equation is to be compared with (35), the elastic cross-section of electron scattered by a pointlike fermion N_{pt}. In this case, using the relation (31), i.e. $q^2 = -2M\nu$ or $\int \mathrm{d}\nu\,\delta(\nu + \frac{q^2}{2M}) = 1$, (35) can be rewritten as

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}(\mathrm{e}+\mathrm{N}_{\mathrm{pt}}\to\mathrm{e}+\mathrm{N}_{\mathrm{pt}})}{\mathrm{d}Q^2\mathrm{d}\nu} = \frac{\pi\,\sigma_{\mathrm{Mott}}}{EE'}\left[\,1\ +2\,\left(\frac{-q^2}{4M^2}\right)\tan^2\frac{\theta}{2}\right]\delta\left(\nu + \frac{q^2}{2M}\right). \tag{10.45}$$

Comparing (44) with (45), we deduce that $W_{1,2}(q^2,\nu)$ tend to the following simplest expressions of the elastic scattering by a pointlike nucleon:

$$\frac{W_2(q^2,\nu)}{M} \longrightarrow \delta\left(\nu + \frac{q^2}{2M}\right) \ , \ \frac{W_1(q^2,\nu)}{M} \longrightarrow \left(\frac{-q^2}{4M^2}\right)\delta\left(\nu + \frac{q^2}{2M}\right). \tag{10.46}$$

It is possible to extract $W_1(q^2,\nu)$ and $W_2(q^2,\nu)$ by plotting data of $\mathrm{d}\sigma_{\mathrm{in}}$ given by (44) as a function of $\tan^2\frac{\theta}{2}$ for fixed q^2, exactly as in the case of $\mathrm{d}\sigma_{\mathrm{el}}$ with $G_{\mathrm{E}}(q^2)$ and $G_{\mathrm{M}}(q^2)$ discussed previously in (34).

On the other hand, we can always write $W_{1,2}(q^2,\nu) \equiv W_{1,2}(q^2,x)$ as functions of the Bjorken variable $x \equiv -q^2/2M\nu$ and q^2. The invariant mass M_X of the final hadronic state is given by $M_X^2 \equiv (p+q)^2 = M^2 + Q^2(1-x)/x$, for fixed momentum transfer Q^2, each value of x may be associated with a specific hadronic final state: $x \approx 1$ corresponds to the few-body quasielastic scatterings, and $x \approx 0$ (large M_X) to multi-particles. Also, at $x \approx 0$, the nucleon–sea is probed.

It was a great surprise in 1968 when, for the first time, the SLAC–MIT experiments showed that at large q^2, the deep inelastic cross-section appeared *much larger than expected*. In other words, the structure functions $W_{1,2}(x, q^2)$ extracted from data are found to behave very differently from the form factors squared $G^2_{E,M}(q^2)$. Indeed, data show that for fixed values of x, when Q^2 varies from 1 to 25 GeV^2, the maximal energies reached at that time, $W_1(x, q^2)$ and $(\nu/M)W_2(x, q^2)$ are practically constant whereas in the same Q^2 range, the $G^2_{E,M}(q^2)$ drop from 1 to 10^{-6}. Nowadays for Q^2 as large as a few thousands of GeV^2 (Fig. 10.7) at HERA, the structure functions remain astonishingly constant or slightly increasing at small x, while the form factors squared decrease from 1 to 10^{-14}.

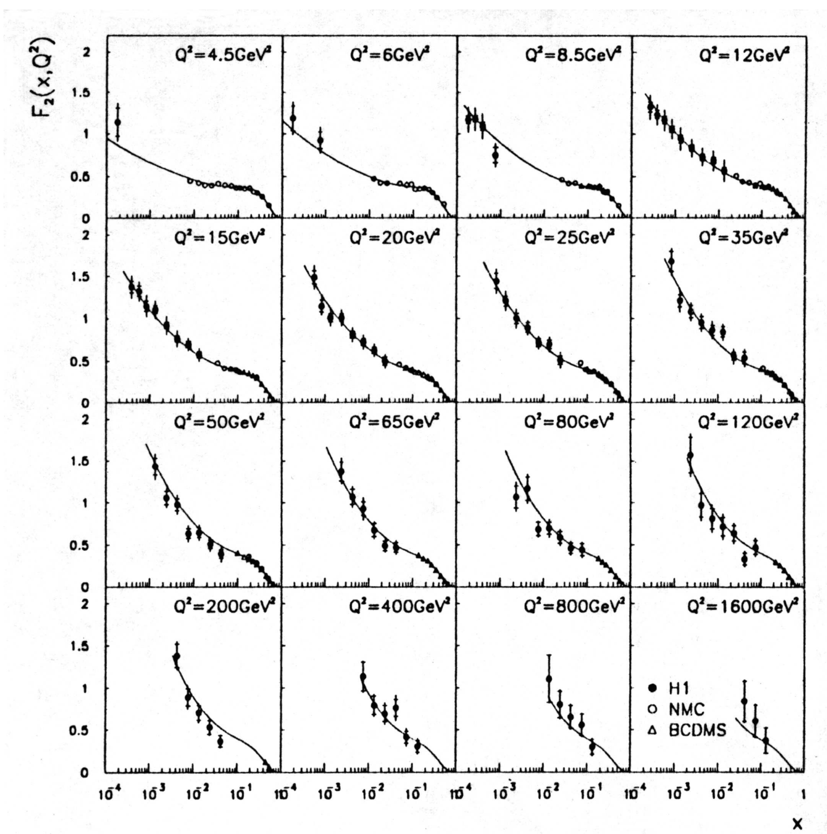

Fig. 10.7. The structure function $F_2(x, q^2)$ (cf. (10.48)) from Dainton, J. in *Proc. Workshop on Deep Inelastic Scattering and QCD* (eds. Laporte, JF. and Sirois, Y.). Editions de l'Ecole Polytechnique, Paris 1995.

10.4.2 Bjorken Scaling and the Feynman Quark Parton

This surprising behavior was in fact already anticipated by Bjorken in 1966. From considerations based on the Gell-Mann quark model current algebra commutators, he discovered the scaling law according to which in the limit

$$-q^2 \equiv Q^2 \to \infty \quad , \qquad \nu \to \infty \quad , \qquad \text{with } \frac{Q^2}{2M\nu} \equiv x \text{ fixed} , \tag{10.47}$$

the structure functions depend only on x :

$$W_1(q^2,\nu) \underset{q^2,\nu\to\infty}{\longrightarrow} F_1(x) \quad , \qquad \frac{\nu}{M} W_2(q^2,\nu) \underset{q^2,\nu\to\infty}{\longrightarrow} F_2(x) . \tag{10.48}$$

The physical content of the Bjorken scaling law (48) lies essentially in the *finite limit* of the structure functions $F_1(x)$ and $F_2(x)$, since one can always write $W_1(q^2,\nu) \equiv F_1(x,q^2)$ and $(\nu/M)W_2(q^2,\nu) \equiv F_2(x,q^2)$. For each fixed value of x, when $-q^2 \to \infty$, the limits of $F_1(x,q^2)$ and $F_2(x,q^2)$ can depend only on x. In principle, they may tend to infinity or zero, the latter possibility is naively expected when we notice that, in a sense, the structure functions represent just an incoherent sum of squared form factors, each tending quickly to zero for large q^2. Bjorken assured us that $F_1(x)$ and $F_2(x)$ are *finite*.

How did Feynman interpret deep inelastic scattering data ? Since the experiments on elastic electron–proton scattering by Hofstadter and his group, who in the 1960s found that the $G_{\mathrm{E}}(q^2)$ and $G_{\mathrm{M}}(q^2)$ form factors decrease as dipole distributions according to (38), it is known that the nucleon has a structure and must be a bound state. But what are its constituents called partons by Feynman, and what is the nature of their interactions ? Two experimental facts obtained from deep inelastic scatterings are crucial: (i) the structure functions are almost independent of q^2, and remarkably do not tend to zero as $q^2 \to \infty$; (ii) the $\tan^2\frac{\theta}{2}$ term is present in (44), i.e. $W_1(q^2,x) \to F_1(x) \neq 0$. We recognize that (i) hints at a loosely bound pointlike parton probed by the virtual photon whereas (ii) suggests that this pointlike constituent is a fermion [remember the second remark (b) after the Rosenbluth formula]. So pointlike quarks naturally emerge from these observations as fundamental constituents of matter. As will be discussed later, the loosely bound parton is a consequence of the QCD asymptotic freedom.

On the other hand, if the proton is a bound state of $\mathrm{n}\pi^+$ or $\Lambda\mathrm{K}^+$ for instance, then $W_j(q^2,\nu)$, with $j = 1, 2$, would be strongly dependent on q^2 (since the constituents n and Λ, unlike the partons, are themselves composite objects like p), and $F_1(x)$ would vanish (since the photon would probe the spinless 'constituent' meson π^+ or K^+ of the proton). Also, the quark *color* degrees of freedom get their dramatic confirmation in the total cross-section $\sigma(\mathrm{e}^+ + \mathrm{e}^- \to \text{hadrons})$, and in the decay rate $\Gamma(\tau \to \nu_\tau + \text{hadrons})$, to mention only two examples (Sect. 7.5 and Chaps. 13–14).

Let us assume that the constituents of nucleons are *valence* quarks $\mathrm{u_v}$, $\mathrm{d_v}$, linked by gluons through non-Abelian QCD interaction and surrounded by pairs of quark–antiquark referred to as the nucleon *sea*. Besides the light quark pairs $\overline{\mathrm{u}}_\mathrm{s}, \mathrm{u}_\mathrm{s}$ and $\overline{\mathrm{d}}_\mathrm{s}, \mathrm{d}_\mathrm{s}$, the sea may also contain strange $\overline{\mathrm{s}}, \mathrm{s}$ and charm $\overline{\mathrm{c}}, \mathrm{c}$ pairs. We distinguish the valence $\mathrm{u}_\mathrm{v}, \mathrm{d}_\mathrm{v}$ from the sea $\mathrm{u}_\mathrm{s}, \mathrm{d}_\mathrm{s}$ and denote their sums as u and d : $\mathrm{u} = \mathrm{u}_\mathrm{v} + \mathrm{u}_\mathrm{s}$, $\mathrm{d} = \mathrm{d}_\mathrm{v} + \mathrm{d}_\mathrm{s}$. When this hypothesis is confronted with experiments, all the data are excellently described: the partons turn out to be quarks, antiquarks, and gluons.

Indeed for large q^2, the photon penetrates more and more deeply into the nucleon and strikes the nucleon partons. It interacts on the one hand with quarks u, d, antiquarks $\overline{\mathrm{u}}, \overline{\mathrm{d}}$ and with other flavored pairs $\overline{\mathrm{q}}\mathrm{q}$ of the sea; on the other hand, like the other gauge bosons W and Z, the photon is insensitive to gluons. The interaction between partons becomes weaker and weaker, at large q^2, the struck quarks (antiquarks) behave as if they were loosely bound, i.e. almost noninteracting or free. They interact softly with the remaining partons, so that when hit by photons, the outgoing partons materialize as jets of hadrons collinear with the directions of the struck partons.

On the other hand, if quarks are not probed by high q^2, ν photons, they are strongly interacting and firmly bound in the hadron. This strange behavior of quarks – their mutual interactions are stronger at low energy than at high energy – can only be understood by the asymptotic freedom of non-Abelian QCD. This property will be discussed in Chap. 15.

Of course we do not observe partons in the final state, but only hadrons. Somehow the scattered and unscattered partons must recombine to form hadrons. The basic assumption is that the collision occurs in two steps. First, a parton is hit during the collision time interval t_1 defined by the energy transfer i.e. $t_1 \sim \hbar/\nu$. At a much later time t_2, the partons recombine to form hadrons of invariant mass M_X, i.e. $t_2 \geq \hbar/M_X$, or in the laboratory frame $t_2 \geq \hbar\nu/M_X^2$. Since $M_X^2 \sim 2M\nu$, we have $t_2 \geq \hbar/2M$ and $t_2 \gg t_1$ is equivalent to $\nu \gg M$ which is the Bjorken limit. Scaling implies that during such a rapid scattering, interactions among the partons are negligible, they are nearly 'free'. The cross-section depends foremost on the dynamics of the first step and very weakly on the complexities of recombination into hadrons in the second step. High-energy deep inelastic experimental data are described as an *incoherent* sum of elastic electron–quarks (or electron–antiquarks) scatterings. Only incoherent additions take place, because the struck quarks (antiquarks) are noninteracting and independent of each other. As we will see later, the structure functions $W_{1,2}$ are essentially total cross-sections, so in a constituent model, it is natural that cross-sections should be independently additive without interference.

To go further, let us denote by z_j the nucleon fractional momentum carried away by a parton j, i.e. the parton momentum k_j^μ is equal to $z_j P^\mu$ where P^μ is the nucleon momentum (Fig. 10.6c). For on-shell partons, $(q + z_j P)^2 = m_j^2$, and we have

$$q^2 + 2M\nu\, z_j = 0 \quad \Longrightarrow z_j = x \equiv \frac{Q^2}{2M\nu}\,.$$

The relation $z_j = x$ allows us to interpret the Bjorken variable x as the fraction of the nucleon momentum carried away by partons. Since the invariant mass of the unobserved hadrons is larger than the nucleon mass, $p_X^2 \equiv (P+q)^2 \geq M^2 \implies q^2 + 2M\nu \geq 0$, we get $0 \leq x \leq 1$. The parton mass m_j is equal to Mz_j, since $m_j^2 \equiv k_j^2 = (z_j P)^2 = M^2 z_j^2$. The cross-section $\mathrm{d}\sigma_j$ of electron scattered by a quark (or antiquark) j of charge e_j (in units of $e > 0$) can be obtained from (35) in which the mass M is to be replaced by $m_j = Mz_j$ together with an overall common factor e_j^2. From (45), we get

$$\frac{\mathrm{d}\sigma_j}{\mathrm{d}Q^2\mathrm{d}\nu} = \frac{\pi\,\sigma_{\mathrm{Mott}}}{EE'}\left[e_j^2 + 2\,e_j^2\left(\frac{Q^2}{4m_j^2}\right)\tan^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2m_j}\right) . \tag{10.49}$$

The contribution of each parton j to the structure functions is immediately recognized by comparing (44) and (49). We call them $w_1(j)$ and $w_2(j)$:

$$\begin{aligned} w_1(j) &= Me_j^2\frac{Q^2}{4m_j^2}\delta\left(\nu - \frac{Q^2}{2m_j}\right) = e_j^2\frac{Q^2}{4Mz_j^2}\delta\left(\nu - \frac{Q^2}{2Mz_j}\right) , \\ w_2(j) &= Me_j^2\delta\left(\nu - \frac{Q^2}{2Mz_j}\right) . \end{aligned} \tag{10.50}$$

Since each parton contributes incoherently to the cross-section, to obtain the structure functions $W_{1,2}(q^2,\nu)$, we simply add up the $w_{1,2}(j)$, each being weighted by the probability $\mathcal{F}_j(z_j)$ for the parton j to have a four-momentum $z_j P^\mu$, and finally we integrate over the whole range of z_j. Note that

$$\delta\left(\nu - \frac{Q^2}{2Mz_j}\right) = \delta\left[\frac{\nu}{z_j}(z_j - \frac{Q^2}{2M\nu})\right] = \frac{z_j}{\nu}\delta(z_j - x) . \tag{10.51}$$

We identify partons as quarks or antiquarks in the construction of $W_{1,2}(q^2,\nu)$. On the other hand, the partonic gluons which are insensitive to electroweak interactions do not contribute to the structure functions. The probabilities $\mathcal{F}_j(z_j = x)$ for finding the partons j are nothing but the distributions in the nucleon of $u(x), d(x), s(x), \overline{u}(x), \overline{d}(x), \overline{s}(x)$ [may be $c(x), \overline{c}(x)$]. For the moment, we keep the generic $\mathcal{F}_j(z_j = x)$ and get from (50) and (51):

$$\begin{aligned} W_1(q^2,\nu) &= \sum_j \int_0^1 \mathrm{d}z_j \mathcal{F}_j(z_j) w_1(j) = \sum_j \int_0^1 \mathrm{d}z_j \mathcal{F}_j(z_j)\frac{e_j^2 Q^2}{4Mz_j^2}\frac{z_j}{\nu}\delta(z_j - x) \\ &= \sum_j \int_0^1 \mathrm{d}z_j \mathcal{F}_j(z_j) e_j^2 \frac{x}{2z_j}\delta(z_j - x) = \sum_j \frac{e_j^2 \mathcal{F}_j(x)}{2} \equiv F_1(x) , \\ W_2(q^2,\nu) &= \sum_j \int_0^1 \mathrm{d}z_j \mathcal{F}_j(z_j) w_2(j) = \sum_j \int_0^1 \mathrm{d}z_j \mathcal{F}_j(z_j) e_j^2 M \frac{z_j}{\nu}\delta(z_j - x) \\ &= \frac{M}{\nu}\sum_j e_j^2\, x\, \mathcal{F}_j(x) \implies \frac{\nu}{M} W_2(q^2,\nu) = \sum_j e_j^2\, x\, \mathcal{F}_j(x) \equiv F_2(x). \end{aligned} \tag{10.52}$$

From the above equation, we deduce directly the Callan–Gross relation

$$2xF_1(x) = F_2(x) \tag{10.53}$$

which expresses the simple fact that the parton hit by the photon is a spin-$1/2$ object; a spinless parton would give $F_1(x) = 0$. Since by definition, partons are pointlike and devoid of form factors, we easily understand why the structure functions are q^2-independent, at least when QCD effects are ignored.

Let us mention that QCD corrections induce a smooth logarithmic q^2 dependence for the structure functions in a definite way. As q^2 increases, $F_{1,2}(x, q^2)$ slightly increase for small x. For large x the tendency is reversed, the structure functions become smaller. Only non-Abelian gauge theory with its asymptotic freedom property can offer such a behavior. In other words, the Bjorken scaling law is predicted to be smoothly violated; such a violation is indeed experimentally confirmed (see Fig. 10.7). The message can hardly be clearer: first, one needs spin-$1/2$ pointlike constituents to describe the Bjorken scaling; second, the scaling violation is predicted in a clear-cut way. The observation of the QCD effects on electromagnetic and weak processes (Chaps. 14 and 16) is one of the great triumphs of the standard model.

The Callan–Gross relation may also be interpreted as follows. If we look at the diagram Fig. 10.6b for $\mathrm{e} + \mathrm{N} \to \mathrm{e} + X$ as a two-step reaction $\mathrm{e} \to \mathrm{e} + \gamma^*$, $\gamma^* + \mathrm{N} \to X$, where γ^* is a virtual photon, then we realize that the structure functions $W_{1,2}(q^2, \nu)$ represent in fact $\sigma_{\mathrm{tot}}(\gamma^* + \mathrm{N})$, the total photoabsorption cross-sections by nucleon. In the nucleon rest frame, ν is just the energy of the virtual photon, and q^2 is the square of its invariant mass. With a nonzero mass, the virtual photon not only has two transverse polarizations $\varepsilon^\mu_\pm(q)$, but also a longitudinal polarization $\varepsilon^\mu_{\mathrm{L}}(q)$, and the corresponding total cross-sections are given by

$$\begin{aligned}
&\sigma_\pm(q^2, \nu) = K\ \varepsilon^\mu_\pm(q)\varepsilon^{*\rho}_\pm(q) W_{\mu\rho}(q^2, \nu)\ ,\\
&\sigma_{\mathrm{L}}(q^2, \nu) = K\ \varepsilon^\mu_{\mathrm{L}}(q)\varepsilon^{*\rho}_{\mathrm{L}}(q) W_{\mu\rho}(q^2, \nu)\ .
\end{aligned} \tag{10.54}$$

Following conventions appropriate for real photons, the kinematic flux factor denoted by K is given by $K = 8\pi^2\alpha/(q^2 + 2M\nu)$. With the definitions

$$p_\mu = (M, 0, 0, 0)\ ;\ q_\mu = (\nu, 0, 0, q_3 = \sqrt{Q^2 + \nu^2})\ ;\ \varepsilon^\mu_\pm = \frac{-1}{\sqrt{2}}(0, 1, \pm i, 0)\ ;$$

$$\varepsilon^\mu_{\mathrm{L}} = \frac{1}{\sqrt{Q^2}}(\sqrt{Q^2 + \nu^2}, 0, 0, \nu)\ ;\ \varepsilon^\mu_\pm \varepsilon^{*\rho}_\pm g_{\mu\rho} = -1\ ;\ \varepsilon^\mu_{\mathrm{L}} \varepsilon^{*\rho}_{\mathrm{L}} g_{\mu\rho} = +1,$$

we get

$$\varepsilon^\mu_\pm \varepsilon^{*\rho}_\pm T^1_{\mu\rho} = 1\ ,\ \varepsilon^\mu_\pm \varepsilon^{*\rho}_\pm T^2_{\mu\rho} = 0\ ,\ \varepsilon^\mu_{\mathrm{L}} \varepsilon^{*\rho}_{\mathrm{L}} T^1_{\mu\rho} = -1\ ,\ \varepsilon^\mu_{\mathrm{L}} \varepsilon^{*\rho}_{\mathrm{L}} T^2_{\mu\rho} = \frac{Q^2 + \nu^2}{Q^2}\ .$$

Then (54) becomes

$$\sigma_{\mathrm{T}}(q^2,\nu) \equiv \frac{1}{2}[\sigma_+(q^2,\nu)+\sigma_-(q^2,\nu)] = K\,W_1(q^2,\nu)\;,$$
$$\sigma_{\mathrm{L}}(q^2,\nu) = K\,\left[-W_1(q^2,\nu)+\left(1+\frac{\nu^2}{Q^2}\right)W_2(q^2,\nu)\right]\;. \tag{10.55}$$

In the Bjorken limit where $Q^2\;,\;\nu\to\infty$,

$$\sigma_{\mathrm{T}} \to K\,F_1(x)\quad,\quad \sigma_{\mathrm{L}} \to K\,\frac{F_2(x)-2xF_1(x)}{2x}\;,$$
$$R \equiv \frac{\sigma_{\mathrm{L}}(q^2,\nu)}{\sigma_{\mathrm{T}}(q^2,\nu)} \to \frac{F_2(x)-2xF_1(x)}{2xF_1(x)}\;. \tag{10.56}$$

The Callan–Gross relation which is equivalent to $R \equiv \sigma_{\mathrm{L}}/\sigma_{\mathrm{T}} \to 0$ agrees with data. A spinless parton would give $F_1(x)=0$ or $R\to\infty$. Such a possibility is certainly ruled out by experiments. The reason why R is not identically equal to zero is that the photon also hits the nucleon sea $\overline{\mathrm{q}}\mathrm{q}$ which is presumably spinless. The fact that $R \ll 1$ implies that the sea contribution is globally much smaller than the quark contributions.

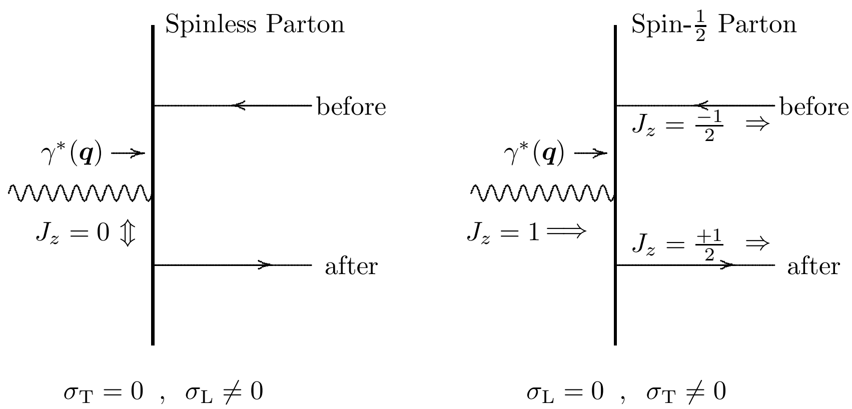

Fig. 10.8. Absorption of a virtual photon by a parton in the Breit frame

We may understand the relation $\sigma_{\mathrm{L}}\to 0$ by considering the helicities of photon and parton in the Breit frame (Fig. 10.8). Indeed in this frame, the three-momenta of the photon and the incoming parton are collinear and opposite, moreover the parton momenta before and after the collision are reversed and exactly equal. By angular momentum conservation, a spinless parton cannot absorb the transverse components of a photon (with its spin $J_z=\pm1$ aligned along its three-momentum vector $\boldsymbol{q}$) but may absorb its longitudinal component (corresponding to $J_z=0$); in other words, with spinless parton, one gets $\sigma_{\mathrm{T}}=0$ and $\sigma_{\mathrm{L}}\neq 0$. On the other hand, along the $\boldsymbol{q}$ axis, an incoming spin-$1/2$ parton with its helicities $\mp 1/2$ can catch a ± 1 photon's spin and go away with $\pm 1/2$ helicities. In this Breit frame, the parton helicity conservation implies a change by ± 1 unit along the $\boldsymbol{q}$ axis before and after the collision that only a transverse photon can satisfy, hence $\sigma_{\rm L} = 0$ and $\sigma_{\rm T} \neq 0$.

We rewrite the probabilities $\mathcal{F}_j(z_j = x)$ in (52) as the distributions of quarks and antiquarks in the nucleon:

$$F_2^{\rm p}(x) = x\left\{\frac{4}{9}[u(x)+\overline{u}(x)] + \frac{1}{9}[d(x)+\overline{d}(x)+s(x)+\overline{s}(x)]+\cdots\right\}\ , \quad (10.57)$$

where the dots stand for possible charm quarks $c(x), \overline{c}(x)$ in the nucleon sea. Invariance of the nucleon properties to isospin rotations implies that the same $u(x), d(x)$ quark distributions in the proton can be used for the neutron. The isospin symmetry by up $\leftrightarrow$ down quarks interchange in proton $\leftrightarrow$ neutron makes the $u^{\rm p}(x)$ [respectively $d^{\rm p}(x)$] distribution in the proton equal to the $d^{\rm n}(x)$ [respectively $u^{\rm n}(x)$] distribution in the neutron:

$$u^{\rm p}(x) = d^{\rm n}(x) \equiv u(x)\ \ ,\ \ d^{\rm p}(x) = u^{\rm n}(x) \equiv d(x)\ . \quad (10.58)$$

From (58) we have

$$F_2^{\rm n}(x) = x\left\{\frac{4}{9}[d(x)+\overline{d}(x)] + \frac{1}{9}[u(x)+\overline{u}(x)+s(x)+\overline{s}(x)]+\cdots\right\}\ . \quad (10.59)$$

Since the nucleon has neither strangeness nor charm, one has

$$\int_0^1 {\rm d}x[s(x)-\overline{s}(x)] = \int_0^1 {\rm d}x[c(x)-\overline{c}(x)] = 0\ . \quad (10.60)$$

Similarly, since the proton and neutron charges are respectively 1 and 0,

$$\begin{aligned}\int_0^1 {\rm d}x\left\{\frac{2}{3}[u(x)-\overline{u}(x)] - \frac{1}{3}[d(x)-\overline{d}(x)]\right\} &= 1\ ,\\ \int_0^1 {\rm d}x\left\{\frac{2}{3}[d(x)-\overline{d}(x)] - \frac{1}{3}[u(x)-\overline{u}(x)]\right\} &= 0\ ,\end{aligned} \quad (10.61)$$

from which

$$\int_0^1 {\rm d}x[u(x)-\overline{u}(x)] = 2 \quad \text{and} \quad \int_0^1 {\rm d}x[d(x)-\overline{d}(x)] = 1\ . \quad (10.62)$$

We now remark that the quark and antiquark distributions also appear in deep inelastic neutrino scattering by nucleon (Chap. 12). Electromagnetic and weak inclusive reactions are related, thus the quark parton model can also be tested by neutrinos as projectiles. From experimental data on $F_2^{\rm p,n}(x)$

(by electromagnetic scattering) and on $F_{2,3}^{\nu}(x)$ (by neutrino scattering), we can extract the distributions $u(x), d(x), s(x), \overline{u}(x), \overline{d}(x)$, and $\overline{s}(x)$ which are constrained by the sum rules (60)–(62).

Indirect evidence for gluons emerges from the observation that the nucleon total momentum must be shared by all of its contituents, gluons included. Since each parton has a momentum $z_j P^\mu$, one must have the sum rule

$$\sum_j \int_0^1 (z_j P^\mu)\mathcal{F}_j(z_j)\mathrm{d}z_j = P^\mu \implies \sum_j \int_0^1 x\mathcal{F}_j(x)\mathrm{d}x = 1 \; . \tag{10.63}$$

To estimate the quark and antiquark contributions to the above sum rule, we now replace in (63) $\mathcal{F}_j(x)$ by the $\mathrm{q}(x), \overline{\mathrm{q}}(x)$ distributions. With (57) and (59), their contributions can be written as

$$F_2^{\mathrm{p}}(x) + F_2^{\mathrm{n}}(x) = \frac{5}{9}x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x)\right] + \frac{2}{9}x\left[s(x) + \overline{s}(x)\right] \; . \tag{10.64}$$

The sum $F_2^{\mathrm{p}}(x) + F_2^{\mathrm{n}}(x) \equiv F_2^{I=0}(x)$ is obtained from data of electron scattering by an isoscalar target (deuteron). We rewrite (64) under the integrated form

$$\frac{9}{5}\int_0^1 \mathrm{d}x F_2^{I=0}(x) = \int_0^1 \mathrm{d}x\; x\left[u(x) + d(x) + \overline{u}(x) + \overline{d}(x) + \frac{2}{5}\left[s(x) + \overline{s}(x)\right]\right] .$$

The integration of the left-hand side of (64) is found to be 0.245 ± 0.02. Provided that $\frac{3}{5}x\left[s(x) + \overline{s}(x)\right] \ll 1$, which is plausible for the sea, the above equation may be written as

$$\sum_j \int_0^1 \mathrm{d}x\; x\left[q_j(x) + \overline{q}_j(x)\right] \approx \frac{9}{5}\,(0.245 \pm 0.02) = 0.44 \pm 0.02 \; .$$

Comparing the above equation with (63), we find that the remaining $(56\mp2)\%$ of the nucleon momentum is carried away not by quarks and antiquarks but by objects insensitive to photons, and gluons are natural candidates.

Finally, in terms of the scaling structure function $F_2(x)$ and the ratio $R \equiv \sigma_{\mathrm{L}}/\sigma_{\mathrm{T}} = -1 + F_2(x)/2xF_1(x)$, the deep inelastic cross-section (44) can be rewritten with a new variable $y \equiv \nu/E = (E-E')/E$ which evidently satisfies $0 \le y \le 1$. This energy loss variable y is also useful in the description of deep inelastic neutrino scattering (Chap. 12). Since $\mathrm{d}Q^2\mathrm{d}\nu = 2ME\,\nu\,\mathrm{d}x\,\mathrm{d}y$, we have

$$\begin{aligned}\frac{\mathrm{d}\sigma(\mathrm{e} + \mathrm{N} \to \mathrm{e} + X)}{\mathrm{d}x\mathrm{d}y} &= \frac{(2ME\,\nu)\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}\nu} = \frac{4\pi\alpha^2\, s}{Q^4}\left\{F_2(x)(1-y) + xF_1(x)y^2\right\} \\ &= \frac{4\pi\alpha^2\, s}{Q^4}F_2(x)\left\{(1-y) + \frac{y^2}{2(1+R)}\right\} \; .\end{aligned} \tag{10.65}$$

The above result is obtained by using the identity $2E(1-y)\sin^2(\theta/2) = Mxy$ and neglecting $M^2 \ll s \approx 2ME$. Since $Q^2 = s\,xy$, (65) may be written as

$$\frac{\mathrm{d}\sigma(\mathrm{e}+\mathrm{N}\to\mathrm{e}+X)}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\frac{F_2(x)}{x}\left\{(1-y)+\frac{y^2}{2(1+R)}\right\}\,. \tag{10.66}$$

For each fixed value of x, the $\mathrm{d}\sigma/\mathrm{d}y$ (or $\mathrm{d}\sigma/\mathrm{d}Q^2$) distribution allows us to extract the structure function $F_2(x)$. When the violation of the Bjorken scaling law is incorporated, (65) is usually written with $F_2(x,q^2)$, its q^2 dependence is smoothly logarithmic as predicted by QCD and experimentally confirmed.

In summary, the photon is a powerful probe of the ultimate structure of matter. The main reason for the photon usefulness is that in the one-photon exchange of deep inelastic scattering, via $\ell^- \to \ell^- + \gamma^*$, the invariant mass squared q^2 and the energy ν of the virtual photon γ^* can be varied when the latter strikes the hadronic constituents to explore their nature. We have seen the crucial role of high q^2, ν in elastic and deep inelastic scatterings. In pion–nucleon collision taken as an example of the more general hadron–hadron scattering, one cannot take advantage of the well-known photon exchanged mechanism of the electromagnetic interaction, i.e. first we make two unknown hadrons collide on each other, second we can only vary the pion energy but we cannot change its mass which must be m_π. In this context, hadron–hadron collisions are not the right way to discover the hadronic constituents. On the other hand, in electromagnetic e–N scattering, only the strong interaction of the struck hadron N is unknown. The more familiar photon, with both its invariant mass and its energy increasing, is capable of revealing the hadronic layers. Once the quarks and gluons are fully recognized as the fundamental constituents of matter, hadron–hadron collisions can be interpreted as quark–quark, quark–gluon, gluon–gluon reactions in a subsequent step. Further evidence for quarks is provided by the neutrino deep inelastic collision (Chap. 12) and by the $\mathrm{e}^+ + \mathrm{e}^-$ annihilation into hadrons. The total cross-section of the latter process fits so well with the $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{q} + \overline{\mathrm{q}}$ (see Sect. 7.5 and Chap. 14 where the QCD radiative corrections are included) that the reality of color quarks can hardly be denied.

Problems

10.1 OPE Yukawa nucleon potential. Show that the matrix element of the one pion exchange (OPE) between the two nucleons 1,2 can be written as the difference of the direct and the exchange terms:

$$\mathcal{M} = g^2_{\pi\mathrm{NN}}\frac{[\overline{u}(p_1', s_1')\,\gamma_5\,\tau_j\,u(p_1,s_1)]\,[\overline{u}(p_2', s_2')\,\gamma_5\,\tau_j\,u(p_2,s_2)]}{(p_1-p_1')^2 - m_\pi^2}$$
$$-\text{ exchange term }[(p_1', s_1') \Leftrightarrow (p_2', s_2')].$$

Why the minus sign? Show that $\overline{u}(p_1', s_1')\,\gamma_5\,u(p_1,s_1) = \chi_1^{'\dagger}(\boldsymbol{\sigma}\cdot\boldsymbol{q})\chi_1 \stackrel{\mathrm{def}}{=} \boldsymbol{\sigma}_1\cdot\boldsymbol{q}$ where $\boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_1'$ in the center-of-mass system. χ is the standard two-component Pauli spinor describing the $\pm 1/2$ spin states of the nucleon (Chap. 3). Then the nonrelativistic limit of the direct term is

$$\mathcal{M}_{\mathrm{dir}}(\boldsymbol{q}) = \left[\frac{-g^2_{\pi\mathrm{NN}}\,(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})}{|\boldsymbol{q}|^2 + m^2_\pi}\right] .$$

The nucleon–nucleon potential $V_{\mathrm{dir}}(\boldsymbol{x})$ is obtained from the Fourier transform of $V_{\mathrm{dir}}(\boldsymbol{q})$, the latter is related to the invariant nucleon–nucleon scattering matrix element $\mathcal{M}$ by $V_{\mathrm{dir}}(\boldsymbol{q}) = \mathcal{M}_{\mathrm{dir}}(\boldsymbol{q})/4M^2_{\mathrm{N}}$ in the nonrelativistic limit

$$V_{\mathrm{dir}}(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3 q\ \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}\frac{\mathcal{M}_{\mathrm{dir}}(\boldsymbol{q})}{4M^2_{\mathrm{N}}} .$$

Explain the origin of the denominator $4M^2_{\mathrm{N}}$. Show that ($r = |\boldsymbol{x}|$)

$$V_{\mathrm{dir}}(\boldsymbol{x}) = \frac{g^2_{\pi NN}}{4\pi}\frac{m^2_\pi}{12M^2_{\mathrm{N}}}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\Big\{\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\frac{\mathrm{e}^{-m_\pi r}}{r} +$$
$$\left[\frac{3\sigma_1 \cdot \boldsymbol{x}\sigma_2 \cdot \boldsymbol{x}}{r^2} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\right]\frac{\mathrm{e}^{-m_\pi r}}{r}\left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right)\Big\} .$$

10.2 Normalization of $F_\pi(0)$. Show that at $q^2 = 0$, the pion form factor satisfies $F_\pi(0) = 1$. Generalize it to the nucleon case and get $F^{\mathrm{p}}_1(0) = 1,\ F^{\mathrm{n}}_1(0) = F^{\mathrm{n}}_2(0) = 0$.

10.3 Meson form factors. In terms of the electromagnetic form factor $F_\pi(q^2)$, write the amplitude of the reaction $\mathrm{e}^+(p') + \mathrm{e}^-(p) \to \pi^+(k') + \pi^-(k)$. The timelike photon exchange (Fig. 10.1) is in the channel of the Mandelstam variable $s \equiv q^2 = (k' + k)^2 > 0$. In the center-of-mass (cm) system of $\mathrm{e}^+\mathrm{e}^-$, show that the cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\mathrm{cm}}} = \frac{\alpha^2|F_\pi(s)|^2}{8\,s}\left(1 - \frac{4m^2_\pi}{s}\right)^{3/2}(1 - \cos^2\theta_{\mathrm{cm}})\ ,\ \boldsymbol{p}\cdot\boldsymbol{k} = |\boldsymbol{p}|\cdot|\boldsymbol{k}|\cos\theta_{\mathrm{cm}} .$$

So σ has a broad peak at $s \approx m^2_{\rho^0}$. Compare this result with the formula (36) of the reaction $\mathrm{e}^-(p) + \pi^\pm(k) \to \mathrm{e}^-(p') + \pi^\pm(k')$, the spacelike photon exchange is now in the $t \equiv (k' - k)^2 \le 0$ channel. What do we expect of the cross-sections $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{K}^+ + \mathrm{K}^-\ ,\ \mathrm{D}^+ + \mathrm{D}^-\ ,\ \mathrm{B}^+ + \mathrm{B}^-$?

10.4 β-decay of $\pi^\pm$. The conserved vector current (CVC) relates the (isospin 1) electromagnetic current J^μ_{em} to V^μ, the *flavor-conserving vector* part of the $V^\mu - A^\mu$ charged current in the $\mathrm{d} \to \mathrm{u}$ transition occurred for example in neutron β decay. Show that

$$\left\langle\pi^0(p')\,|\,V^\mu\,|\,\pi^\pm(p)\right\rangle = \sqrt{2}(p' + p)^\mu\,F_\pi(q^2) .$$

Compute the rate $\pi^+ \to \pi^0 + \mathrm{e}^+ + \nu_\mathrm{e}$. Generalize to $\overline{\mathrm{K}}^0 \to \mathrm{K}^- + \mathrm{e}^+ + \nu_\mathrm{e}$. Why is the factor $\sqrt{2}$ absent in the K case ?

10.5 ρ^0 decay. By dimensional argument, show that the couplings between $\pi^\pm - W^\pm$ and $\gamma - \rho^0$ have the dimension of [mass]^2. Write the amplitudes $\pi^+ \to \mathrm{e}^+ + \nu_\mathrm{e}$ and $\rho^0 \to \mathrm{e}^+ + \mathrm{e}^-$ in terms of the decay constants f_π and f_{ρ^0}; both have the dimension of [mass] (Fig. 10.3a, b). The former amplitude is proportional to the electron mass, whereas the latter is not. Therefore $\Gamma(\pi^+ \to \mathrm{e}^+ + \nu_\mathrm{e}) \ll \Gamma(\pi^+ \to \mu^+ + \nu_\mu)$. Using CVC, compute the $\rho^+ \to \mathrm{e}^+ + \nu_\mathrm{e}$ rate. Check the formulas (20), (21) of $\rho^0 \to \pi^+ + \pi^-$ and $\rho^0 \to \mathrm{e}^+ + \mathrm{e}^-$.

10.6 Inverse Fourier transform of form factors. The distribution $\rho(r)$ of charge in hadrons can be obtained from the inverse Fourier transform of the corresponding form factors in the nonrelativistic limit ($q^2 \to -|\boldsymbol{q}|^2$). Show that

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{1 - \frac{q^2}{\Lambda^2}} = \frac{\Lambda^2}{4\pi} \frac{\mathrm{e}^{-\Lambda r}}{r} \, , \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \mathrm{e}^{\frac{-q^2}{\Lambda^2}} = \frac{\Lambda^3 \mathrm{e}^{-\frac{\Lambda^2 r^2}{4}}}{8\pi\sqrt{\pi}} \, .$$

10.7 Omnès–Muskhelishvili form factor representation. According to the Watson theorem, we have $F_\pi(s) = |F_\pi(s)| \mathrm{e}^{\mathrm{i}\delta(s)}$, where $\delta(s)$ is the p-wave phase-shift of π–π strong interaction. Write the dispersion relation for $G(s) \equiv \log F_\pi(s)$. Show that

$$F_\pi(s) = \exp \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta(s')}{s'(s'-s)} \mathrm{d}s \, .$$

If the form factor $F_\pi(s)$ has zeros in the complex s plane, to avoid difficulties when taking its logarithm, one may factorize $F_\pi(s) = P(s)\overline{F}_\pi(s)$ where the polynome $P(s)$ contains all the zeros of $F_\pi(s)$. Repeat the analysis with $\overline{F}_\pi(s)$. Consider now the function

$$H(s) \equiv \frac{\log \overline{F}_\pi(s)}{\sqrt{s - 4m_\pi^2}} .$$

First find the imaginary of $H(s)$ in terms of $|F_\pi(s)|$, the latter is measured directly in $\mathrm{e}^+ + \mathrm{e}^- \to \pi^+ + \pi^-$ (Problem 10.3); then write the dispersion relation for $H(s)$.

Suggestions for Further Reading

General reading, form factors, elastic e–N *scattering:*

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