9 The Standard Model of the Electroweak Interaction

In this chapter we describe the unified theory of weak and electromagnetic interactions that is often referred to as 'the standard model' (of the electroweak interaction). It is a non-Abelian gauge theory in which the local phase invariance is hidden, or spontaneously broken, so that the weak gauge forces may acquire a finite range as experimentally observed without sacrificing the renormalizability expected of a physically meaningful theory. We shall first review the developments that have led to the formulation of the current theory. We next describe in some detail the model of the electron and its neutrino, including the identification of the gauge symmetry group, its subsequent spontaneous breaking and the attendant mass generation for the electron and the gauge bosons. Introducing one family of quarks into the model poses no particular problem of principle, but when several families of leptons and quarks enter, extra care must be taken to distinguish between the gauge eigenstates and the mass eigenstates in the fermion sector, which gives rise to several novel and powerful predictions by the theory.

9.1 The Weak Interaction Before the Gauge Theories

Before the advent of the gauge models in the late 1960s, weak transitions have been described by a local two-current interaction originally due to Fermi (Chap. 5) and generalized to its present form by Feynman and Gell-Mann,

$$\mathcal{H}_{\mathrm{weak}} = \frac{G_{\mathrm{F}}}{\sqrt{2}} J'^{\mu} J'^{\dagger}_{\mu} \, . \tag{9.1}$$

Here the Lorentz vector J'_{μ} is a *charged current*, so called because the charge of the particle entering the interaction vertex differs by one unit from that of the particle leaving the vertex. (It equals twice the current J_{μ} to be introduced later in the chapter.) With a current that incorporates both leptons and hadrons,

$$J'_{\mu} = L_{\mu} + H_{\mu} \, , \tag{9.2}$$

the interaction Hamiltonian (1) provides a complete description of the weak processes at low energies, the only energy region where it is regarded as applicable. The only notable exception to this general success is the phenomenon of violation of time-reversal invariance discovered in the neutral K meson system by Christenson, Cronin, Fitch, and Turlay in 1964, which appears to require completely new ideas (Chap. 11).

The coupling constant in (1) is not dimensionless, being given by

$$G_{\mathrm{F}} = 1.166\,39 \times 10^{-5}\,\mathrm{GeV}^{-2}\,. \tag{9.3}$$

Numerically, G_{F} is very small, but having the dimension of the inverse squared mass, it leads to a nonrenormalizable interaction. Corrections beyond the tree-diagram level, which are given by loop graphs with internal particle lines, involve higher powers of G_{F}, or of mass in the denominator, and hence higher powers of momentum in the numerator. This leads to increasingly divergent terms in successive orders of the perturbation theory, which cannot be rearranged so as to be absorbed into a small number of 'bare' parameters and fields to yield a finite theory. The theory is not renormalizable.

But if on dimensional grounds we pose

$$4\sqrt{2}\,G_{\mathrm{F}} = e^2/M_{\mathrm{W}}^2\,, \tag{9.4}$$

the resulting 'mass' $M_{\mathrm{W}} \approx 37$ GeV may be viewed as indicative that the weak interaction might not be inherently feeble after all and its apparent weakness might just come from the presence of a very massive quantum exchanged between interacting particles. Like the electromagnetic current j_μ^{em}, the weak charged current J'_μ is a Lorentz four-vector, and we may use the familiar form of the electromagnetic interaction, $-e\,j_\mu^{\mathrm{em}} A^\mu$, as a model to construct the basic weak interaction, coupling J'_μ to a new *massive charged field* W_μ of mass M_{W}, in the form

$$\mathcal{L}_{\mathrm{weak}} = -\frac{g}{2\sqrt{2}}(J'^\mu W_\mu^\dagger + J'^{\mu\dagger} W_\mu)\,. \tag{9.5}$$

Then to second order, $\mathcal{L}_{\mathrm{weak}}$ will generate (1) as an effective low-energy weak interaction with coupling constant $G_{\mathrm{F}}/\sqrt{2} = g^2/(8M_{\mathrm{W}}^2)$, where M_{W}^2 comes from the W_μ field propagator in the limit of *small momentum*. Even though the new coupling constant is dimensionless, the theory is still not manifestly renormalizable because, as we have seen in the last chapter, the propagator of a massive vector particle reduces (also) to M_{W}^{-2} at *large momentum*, leading to divergent integrals in higher-order diagrams. Nevertheless, such theories can be renormalized provided that gauge invariance holds. Thus, gauge invariance is the key. The problem lies in formulating a gauge theory of weak interactions containing massive gauge fields while preserving renormalizability that can be meshed with the electromagnetic interaction theory into a unified theory of the electroweak interaction.

9.2 Gauge-Invariant Model of One-Lepton Family

In this section, we construct a gauge model for one family of leptons, the electron and its neutrino. It already contains many of the main properties found in the complete unified theory of the electroweak interaction. We first determine the simplest group to be gauged that would give rise to the key features of both the electromagnetic and weak interactions of the leptons. Next, we describe the gauge-invariant model involving leptons and scalar fields. Finally, we discuss in detail how spontaneous symmetry breaking generates masses for both matter and gauge fields.

We can limit ourselves to one family of leptons because, as we have seen previously, there exist electron-type conservation laws: the number of electrons e^- plus the number of electronic neutrinos ν_e, minus the number of the corresponding antiparticles, e^+ and $\overline{\nu}_\mathrm{e}$, is conserved; and similarly for the muon- and tau-type leptons. In the following, we denote the field of a particle by its usual symbol, e.g. ν_e designates the spinor field for the neutrino, and e, the spinor field for the electron.

Any Dirac spinor field can be decomposed into left- and right-handed components

$$\chi(x) = \chi_\mathrm{L}(x) + \chi_\mathrm{R}(x)\,, \tag{9.6}$$

where one defines

$$\chi_\mathrm{L}(x) = a_\mathrm{L}\,\chi(x)\,, \qquad \chi_\mathrm{R}(x) = a_\mathrm{R}\,\chi(x)\,, \tag{9.7}$$

in terms of χ by application of the left and right chiral projection operators

$$\begin{aligned} a_\mathrm{L} &\equiv \tfrac{1}{2}(1-\gamma_5)\,, \\ a_\mathrm{R} &\equiv \tfrac{1}{2}(1+\gamma_5)\,. \end{aligned} \tag{9.8}$$

Note in particular the expressions of their adjoint conjugates

$$\begin{aligned} \overline{\chi}_\mathrm{L} &= \chi_\mathrm{L}^\dagger\gamma_0 = \chi^\dagger\, a_\mathrm{L}\gamma_0 = \overline{\chi}\, a_\mathrm{R}\,, \\ \overline{\chi}_\mathrm{R} &= \chi_\mathrm{R}^\dagger\gamma_0 = \chi^\dagger\, a_\mathrm{R}\gamma_0 = \overline{\chi}\, a_\mathrm{L}\,. \end{aligned} \tag{9.9}$$

As we have seen in Chap. 3, breakup (6) has no Lorentz-invariant meaning when the field is massive. But if on the contrary the mass of the field is zero, either of the two chiral components, which then coincides with a helicity eigenstate, may provide a complete representation of the Lorentz group.

The key experimental fact is that in the spectra of weak decays, such as $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$, $\mu^- \to \mathrm{e}^- + \overline{\nu}_\mathrm{e} + \nu_\mu$, and $\pi^- \to \mu^- + \overline{\nu}_\mu$, only left-handed leptons and right-handed antileptons show up, so that the decay amplitudes can be described in terms of a charged current that involves only the left chiral components of the fields,

$$\begin{aligned} L_\mu(x) &= 2\,\overline{e}_\mathrm{L}(x)\gamma_\mu\nu_\mathrm{eL}(x) + (\text{other lepton types}) \\ &= \overline{e}(x)\gamma_\mu(1-\gamma_5)\nu_\mathrm{e}(x) + \ldots\,. \end{aligned} \tag{9.10}$$

This expression resembles the isovector current introduced in Chap. 6, and suggests that ν_{eL} and e_{L} should be gathered into a two-component vector which can be associated with an SU(2) group, the simplest group having a complex doublet representation. On the other hand, the right chiral components ν_{eR} and e_{R}, which do not interact with any other particles, should be left in one-dimensional representations. But while e_{R} should certainly stay because it has the same nonvanishing charge and mass as e_{L}, the right chiral component of the neutrino ν_{eR} may be immediately dropped because the neutrino is observed left-handed and electrically neutral, and is assumed to be exactly massless.

Therefore, in this model of the electron family, we have as matter fields a doublet, ψ_{L}, and a singlet, ψ_{R}, of an SU(2) group,

$$\psi_{\mathrm{L}} = \begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{L}} \end{pmatrix}, \qquad \psi_{\mathrm{R}} = e_{\mathrm{R}} \,. \tag{9.11}$$

As this group acts nontrivially just on the left chiral fermions, it is sometimes denoted by $\mathrm{SU_L}(2)$ and referred to as the *weak-isospin* group. We will be using a simplified notation in this section, for instance ν for ν_{e}, when no risks of confusion can arise.

9.2.1 Global Symmetry

The free Lagrangian for the (massless) fields in (11) is

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}_{\mathrm{L}}\, \mathrm{i}\gamma^\lambda \partial_\lambda\, \psi_{\mathrm{L}} + \overline{\psi}_{\mathrm{R}}\, \mathrm{i}\gamma^\lambda \partial_\lambda\, \psi_{\mathrm{R}} \\ &= \overline{\nu}_{\mathrm{L}}\, \mathrm{i}\gamma^\lambda \partial_\lambda\, \nu_{\mathrm{L}} + \overline{e}\, \mathrm{i}\gamma^\lambda \partial_\lambda\, e \,.\end{aligned} \tag{9.12}$$

Weak Isospin. By construction, $\mathcal{L}_0$ is invariant to SU(2) transformations

$$U_2(\boldsymbol{\omega}) = \mathrm{e}^{-\mathrm{i}g\omega_i t_i}\,, \tag{9.13}$$

where ω_i, for $i = 1, 2, 3$, are the transformation constant parameters, and t_i is equal to $t_{i\mathrm{L}} = \frac{1}{2}\,\tau_i$ (the usual Pauli matrices) when it operates on ψ_{L}, and to $t_{i\mathrm{R}} = 0$ when it operates on ψ_{R}. For infinitesimal transformations, we write

$$\begin{aligned} &U_2\psi_{\mathrm{L}} = \psi_{\mathrm{L}} + \delta\psi_{\mathrm{L}}, \qquad \delta\psi_{\mathrm{L}} \approx -\mathrm{i}\tfrac{1}{2}\, g\omega_i\tau_i\, \psi_{\mathrm{L}}\,; \\ &U_2\psi_{\mathrm{R}} = \psi_{\mathrm{R}} + \delta\psi_{\mathrm{R}}, \qquad \delta\psi_{\mathrm{R}} = 0\,.\end{aligned} \tag{9.14}$$

In general, the conserved currents associated with continuous global symmetries parameterized by real α_i are defined, as in Chap. 2,

$$j_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\, \frac{\delta\varphi_a}{\delta\alpha_i} + \mathcal{L}\, \frac{\delta x^\mu}{\delta\alpha_i}\,. \tag{9.15}$$

For transformations on internal space, $\delta x^\mu = 0$ and the last term on the right-hand side is absent.

In the present case, we may choose $\alpha_i = g\omega_i$ in (15) so that

$$\frac{\delta\psi_{\mathrm{L}}}{\delta(g\omega_i)} = -\mathrm{i}\frac{\tau_i}{2}\psi_{\mathrm{L}}\,, \qquad \frac{\delta\psi_{\mathrm{R}}}{\delta(g\omega_i)} = 0\,, \tag{9.16}$$

leading to the conserved weak-isospin currents

$$j_i^\mu = \overline{\psi}_{\mathrm{L}}\gamma^\mu \frac{\tau_i}{2}\,\psi_{\mathrm{L}} \quad (i = 1,\, 2,\, 3)\,. \tag{9.17}$$

Of course, they act only on the left chiral component. The corresponding conserved charges are the weak-isopin operators

$$T_i = \int \mathrm{d}^3x\, j_i^0(x) = \int \mathrm{d}^3x\, \psi_{\mathrm{L}}^\dagger \frac{\tau_i}{2}\psi_{\mathrm{L}}\,, \tag{9.18}$$

or, more explicitly,

$$T_1 = \tfrac{1}{2}\int \mathrm{d}^3x\,(\nu_{\mathrm{L}}^\dagger e_{\mathrm{L}} + e_{\mathrm{L}}^\dagger \nu_{\mathrm{L}})\,, \tag{9.19}$$

$$T_2 = -\tfrac{\mathrm{i}}{2}\int \mathrm{d}^3x\,(\nu_{\mathrm{L}}^\dagger e_{\mathrm{L}} - e_{\mathrm{L}}^\dagger \nu_{\mathrm{L}})\,, \tag{9.20}$$

$$T_3 = \tfrac{1}{2}\int \mathrm{d}^3x\,(\nu_{\mathrm{L}}^\dagger \nu_{\mathrm{L}} - e_{\mathrm{L}}^\dagger e_{\mathrm{L}})\,. \tag{9.21}$$

From the usual canonical commutation relations of fermion fields at equal times and the familiar commutation relations of the Pauli matrices,

$$[\tau_i,\, \tau_j] = 2\,\mathrm{i}\epsilon_{ijk}\,\tau_k\,, \tag{9.22}$$

it is a simple exercise to prove that

$$[T_i,\, T_j] = \mathrm{i}\epsilon_{ijk}\,T_k\,, \tag{9.23}$$

or, alternatively,

$$[T^{(+)},\, T^{(-)}] = 2\,T_3\,, \qquad [T^{(\pm)},\, T_3] = \mp\, T^{(\pm)}\,, \tag{9.24}$$

using the definition of the raising and lowering operators $T^{(\pm)} \equiv T_1 \pm \mathrm{i}T_2$.

A mass term of the kind

$$-m\,\overline{e}e = -m(\overline{e}_{\mathrm{R}}e_{\mathrm{L}} + \overline{e}_{\mathrm{L}}e_{\mathrm{R}}) \tag{9.25}$$

would break $\mathrm{SU_L}(2)$ invariance. So that for $\mathcal{L}_0$ to have weak-isospin symmetry, both fermions, the electron and the neutrino, must be massless.

Weak Hypercharge. Evidently, the Lagrangian (12) is also invariant to general phase transformations

$$U(\omega) = \mathrm{e}^{-\mathrm{i}\omega f F},$$

which form a U(1) group for a given quantum number matrix F acting as generator. A constant f, to be identified with a coupling constant, has been separated from the parameter ω. For infinitesimal transformations, we have $U \approx 1 - \mathrm{i}\omega f F$ and

$$\begin{aligned} \psi &\to \psi' = U(\omega)\,\psi \approx \psi + \delta\psi\,, \\ \delta\psi &= -\,\mathrm{i}\omega f\, F\psi\,. \end{aligned} \tag{9.26}$$

The order of the matrix F is given by the dimension of the representation ψ. Substituting the derivatives

$$\frac{\delta\psi_{\mathrm{L}}}{\delta(f\omega)} = -\mathrm{i}F_L\,\psi_{\mathrm{L}}\,, \quad \frac{\delta\psi_{\mathrm{R}}}{\delta(f\omega)} = -\mathrm{i}F_R\,\psi_{\mathrm{R}} \tag{9.27}$$

into (15) yields the conserved current operator

$$j_\mu^{\mathrm{F}} = \overline{\psi}_{\mathrm{L}}\gamma_\mu F_L\psi_{\mathrm{L}} + \overline{\psi}_{\mathrm{R}}\gamma_\mu F_R\psi_{\mathrm{R}}\,, \tag{9.28}$$

which in turn leads to the associated conserved charge operator

$$\boldsymbol{F} = \int \mathrm{d}^3x\, j_0^{\mathrm{F}}(x) = \int \mathrm{d}^3x\, \left(\psi_{\mathrm{L}}^\dagger F_L\psi_{\mathrm{L}} + \psi_{\mathrm{R}}^\dagger F_R\psi_{\mathrm{R}}\right)\,. \tag{9.29}$$

Not any U(1) symmetry of $\mathcal{L}_0$ is compatible with $\mathrm{SU_L}(2)$. When $\boldsymbol{F} = Q$, the electrical charge number operator, we can identify (28) and (29) with the electromagnetic current and charge-number operators, with the usual values of the electrical charges for the neutrino and electron,

$$j_\mu^{\mathrm{em}} = Q_{\mathrm{e}}\,(\overline{e}_{\mathrm{L}}\gamma_\mu e_{\mathrm{L}} + \overline{e}_{\mathrm{R}}\gamma_\mu e_{\mathrm{R}}) = -\overline{e}\gamma_\mu e\,; \tag{9.30}$$

$$Q = \int \mathrm{d}^3x\, j_0^{\mathrm{em}}(x) = -\int \mathrm{d}^3x\, e^\dagger e\,. \tag{9.31}$$

In order for the associated group, $U_{\mathrm{Q}}(1)$, to coexist with $\mathrm{SU_L}(2)$, the charge operator must commute with the isospin operators. But since the two components of the doublet ψ_{L} have different charges, the charge number is clearly not a good quantum number in $\mathrm{SU_L}(2)$. In other words,

$$\begin{aligned} Q &\equiv Q_{\mathrm{L}} + Q_{\mathrm{R}} \\ &= -\int \mathrm{d}^3x\, \psi_{\mathrm{L}}^\dagger \tfrac{1}{2}\,(1-\tau_3)\psi_{\mathrm{L}} - \int \mathrm{d}^3x\, e_{\mathrm{R}}^\dagger e_{\mathrm{R}} \end{aligned} \tag{9.32}$$

does not commute with all $T_i = \frac{1}{2}\,\tau_i$, but rather gives

$$[Q,\, T_i] = [T_3,\, T_i] = \mathrm{i}\epsilon_{3ij}\, T_j\,. \tag{9.33}$$

Therefore, $U_{\mathrm{Q}}(1)$ and $\mathrm{SU_L}(2)$ cannot be simultaneous symmetries of $\mathcal{L}_0$. However, (33) tells us that $\alpha(Q - T_3)$ commutes with T_i for all i and arbitrary constant α, and hence may be regarded as the generator of a U(1) group commuting with $\mathrm{SU_L}(2)$. We choose $\alpha = 2$ and call $2(Q - T_3)$ the *weak-hypercharge* operator, Y, in analogy with the strong hypercharge which was defined for hadrons. The relation

$$Q = T_3 + \tfrac{1}{2}\,Y \tag{9.34}$$

provides then a connection between electricity and the weak interaction for it gives the electric charge of an electron or associated neutrino (or, as it turns out, of any weakly interacting particle) in terms of its weak-isospin z component and its weak hypercharge. An explicit expression for Y may be obtained from (21) and (32) as follows:

$$\begin{aligned} Y &= -\int \mathrm{d}^3x\, \psi_{\mathrm{L}}^{\dagger}(1-\tau_3)\psi_{\mathrm{L}} - 2\int \mathrm{d}^3x\, \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{R}} - \int \mathrm{d}^3x\, \psi_{\mathrm{L}}^{\dagger}\tau_3\psi_{\mathrm{L}} \\ &= -\int \mathrm{d}^3x\, \psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{L}} - 2\int \mathrm{d}^3x\, \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{R}}\,. \end{aligned} \tag{9.35}$$

Identifying this result with (29), $\boldsymbol{F} = Y$, we have for the electron family

$$Y_{\mathrm{L}} = -1 \quad \text{and} \quad Y_{\mathrm{R}} = -2\,. \tag{9.36}$$

For a given isomultiplet, $Y = 2\langle Q - T_3\rangle = 2\langle Q\rangle$, i.e. twice the average charge of the multiplet. The corresponding conserved current

$$j_\mu^{\mathrm{Y}} = Y_{\mathrm{L}}\,\overline{\psi}_{\mathrm{L}}\gamma_\mu\psi_{\mathrm{L}} + Y_{\mathrm{R}}\,\overline{\psi}_{\mathrm{R}}\gamma_\mu\psi_{\mathrm{R}} \tag{9.37}$$

is simply related to the electromagnetic current and the isospin current by

$$j_\mu^{\mathrm{em}} = j_\mu^3 + \tfrac{1}{2}\,j_\mu^{\mathrm{Y}}\,. \tag{9.38}$$

To summarize, the free-lepton Lagrangian $\mathcal{L}_0$ is invariant under the direct product group $\mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ of global transformations

$$\mathrm{SU_L}(2): \qquad U_2(\boldsymbol{\omega}) = \mathrm{e}^{-\mathrm{i}g\omega_i\,\frac{1}{2}\,\tau_i}\,, \tag{9.39}$$

$$\mathrm{U_Y}(1): \qquad U_1(\omega) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}\,g'\omega\,Y}\,, \tag{9.40}$$

where g and g' are eventually identified with coupling constants. We show in Table 9.1 the classification and the assigned quantum numbers of the electron family in $\mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$. It is the symmetry group to be gauged.

Table 9.1. Classification and assigned quantum numbers of the electron family

	T	T_3	Y	Q
$\begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{L}} \end{pmatrix}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	-1	0 -1
e_{R}	0	0	-2	-1

9.2.2 Gauge Invariance

We now proceed to transform the free-particle Lagrangian $\mathcal{L}_0$ into an interacting particle model by applying the gauge invariance principle (Chap. 8). This means four vector boson fields corresponding to the four generators of $\mathrm{SU}(2) \times \mathrm{U}(1)$ will have to be introduced. One of these fields will remain massless to generate the electromagnetic force, and the remaining three will acquire mass via the Higgs mechanism so as to produce the observed short-range weak forces. To make this mechanism work properly, one needs at least one weak-isospin doublet of complex scalar fields, one of which is electrically neutral. These fields interact with each other via self-coupling so that hiding gauge invariance becomes feasible, and also with the electron field in order to eventually give it a mass.

The free-field Lagrangian $\mathcal{L}_0$ is accordingly replaced by its corresponding gauge-invariant form

$$\mathcal{L}_\ell = \overline{\psi}_\mathrm{L}\, \mathrm{i}\gamma^\mu D^\mathrm{L}_\mu\, \psi_\mathrm{L} + \overline{\psi}_\mathrm{R}\, \mathrm{i}\gamma^\mu D^\mathrm{R}_\mu\, \psi_\mathrm{R}\,, \tag{9.41}$$

where the covariant derivatives of fields are

$$D^\mathrm{L}_\mu \psi_\mathrm{L} = \left(\partial_\mu + \mathrm{i}g A_{i\mu}\frac{\tau_i}{2} + \mathrm{i}g' B_\mu \frac{Y_\mathrm{L}}{2}\right)\,\psi_\mathrm{L}\,, \tag{9.42}$$

$$D^\mathrm{R}_\mu \psi_\mathrm{R} = \left(\partial_\mu + \mathrm{i}g' B_\mu \frac{Y_\mathrm{R}}{2}\right)\,\psi_\mathrm{R}\,. \tag{9.43}$$

Here $A_{i\mu}$ are the three vector gauge fields associated with $\mathrm{SU_L}(2)$, and B_μ is the $\mathrm{U_Y}(1)$ gauge field. The dynamics of these fields is contained in the Lagrangian

$$\mathcal{L}_\mathrm{G} = -\frac{1}{4}\, W^i_{\mu\nu} W_i^{\mu\nu} - \frac{1}{4}\, B_{\mu\nu} B^{\mu\nu}\,, \tag{9.44}$$

where

$$W^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\epsilon_{ijk}\, A^j_\mu A^k_\nu\,, \tag{9.45}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{9.46}$$

The Lagrangian $\mathcal{L}_\ell$ and $\mathcal{L}_\mathrm{G}$ are invariant to the $\mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ group of local transformations $U_2[\boldsymbol{\omega}(x)]$ and $U_1[\omega(x)]$ with space-time coordinate-dependent parameters. The transformed fields are given by

$$\psi'_\mathrm{L} = U_2 U_1 \psi_\mathrm{L}\,, \qquad \psi'_\mathrm{R} = U_1 \psi_\mathrm{R}\,; \tag{9.47}$$

$$B'_\mu = B_\mu + \partial_\mu \omega\,; \tag{9.48}$$

$$\boldsymbol{A}'_\mu = U_2 \boldsymbol{A}_\mu U_2^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U_2)U_2^\dagger \approx \boldsymbol{A}_\mu + \partial_\mu \boldsymbol{\omega} + \mathrm{i}g\,[\boldsymbol{A}_\mu, \boldsymbol{\omega}]\,; \tag{9.49}$$

where $\boldsymbol{A}_\mu = \frac{1}{2}\,\tau_i A^i_\mu$ is the Hermitian gauge field matrix.

In order to eventually hide gauge invariance, two complex scalar fields forming an SU(2) doublet (having weak hypercharge called $Y_{\rm H}$) are now introduced:

$$\boldsymbol{\phi} = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} . \tag{9.50}$$

Their dynamics in a self-coupling potential is represented by the gauge-invariant Lagrangian

$$\mathcal{L}_{\rm s} = (D_\mu \boldsymbol{\phi})^\dagger (D^\mu \boldsymbol{\phi}) - V(\boldsymbol{\phi}) \equiv (D_\mu \boldsymbol{\phi})^\dagger (D^\mu \boldsymbol{\phi}) - \mu^2 \boldsymbol{\phi}^\dagger \boldsymbol{\phi} - \lambda\, (\boldsymbol{\phi}^\dagger \boldsymbol{\phi})^2. \tag{9.51}$$

Here the covariant derivatives of the scalar fields are given by

$$\begin{aligned} D_\mu \boldsymbol{\phi} \quad &= \left(\partial_\mu + {\rm i}g\, \boldsymbol{A}_\mu + {\rm i}g'\, B_\mu \frac{Y_{\rm H}}{2} \right) \boldsymbol{\phi} \,, \\ (D_\mu \boldsymbol{\phi})^\dagger &= \partial_\mu \boldsymbol{\phi}^\dagger - {\rm i}g\, \boldsymbol{\phi}^\dagger \boldsymbol{A}_\mu - {\rm i}g'\, B_\mu \frac{Y_{\rm H}}{2} \boldsymbol{\phi}^\dagger \,. \end{aligned} \tag{9.52}$$

Finally, with a view to generating the electron mass, we introduce a gauge-invariant Yukawa coupling between scalars and fermions,

$$\mathcal{L}_{\ell{\rm Y}} = -C_{\rm e} \left[\overline{\psi}_{\rm R} (\boldsymbol{\phi}^\dagger \psi_{\rm L}) + (\overline{\psi}_{\rm L} \boldsymbol{\phi}) \psi_{\rm R} \right] , \tag{9.53}$$

where $C_{\rm e}$, an additional parameter, gives the strength of this coupling. $\mathcal{L}_{\ell{\rm Y}}$ is evidently invariant under $\mathrm{SU_L}(2)$ – which is precisely why we need a doublet of scalars – while its invariance under $\mathrm{U_Y}(1)$ is guaranteed by requiring that $\boldsymbol{\phi}$ have weak hypercharge $Y_{\rm H} = Y_{\rm L} - Y_{\rm R}$, that is, $Y_{\rm H} = 1$. From this assignment and $Q = T_3 + Y/2$, it follows that φ^+ has charge $Q = +1$ and φ^0 has charge $Q = 0$. The presence of such an electrically neutral member in the doublet makes it possible for $\boldsymbol{\phi}$ to develop a $U_{\rm Q}(1)$-invariant expectation value and for one gauge boson to remain massless.

9.2.3 Spontaneous Symmetry Breaking

As we have already discussed in Sect. 8.6, the potential $V(\boldsymbol{\phi})$ with positive λ and negative μ^2 has minima, $(\partial V/\partial \phi) = 0$, at values of $\boldsymbol{\phi}$ given by

$$|\boldsymbol{\phi}|^2 = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \,, \tag{9.54}$$

so that when the scalar doublet develops a vacuum expectation value

$$\langle 0 \,|\, \boldsymbol{\phi} \,|\, 0 \rangle = \boldsymbol{v} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} , \tag{9.55}$$

with real constant $v = \sqrt{-\mu^2/\lambda}$, we have spontaneous symmetry breakdown. It is clear then that neither T_i nor Y cancels $\boldsymbol{v}$. In particular,

$$T_3\,\boldsymbol{v} = -\tfrac{1}{2}\,\boldsymbol{v}\,,$$
$$Y\boldsymbol{v} = Y_{\mathrm{H}}\,\boldsymbol{v} = \boldsymbol{v}\,;$$

but

$$Q\boldsymbol{v} = (T_3 + \tfrac{1}{2}\,Y)\,\boldsymbol{v} = 0\,. \tag{9.56}$$

Thus, SU(2) and U(1) are completely broken separately, but the product group SU(2) $\times$ U(1) is not: after symmetry breaking there remains a residual symmetry generated by Q. This pattern of symmetry breakdown is described by the reduction equation

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}_{\mathrm{Y}}(1) \to \mathrm{U}_{\mathrm{Q}}(1)\,. \tag{9.57}$$

It proves convenient to reparameterize $\boldsymbol{\phi}$ in polar field variables

$$\boldsymbol{\phi} = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = \exp\left(\frac{\mathrm{i}}{v}\sum \xi_i T_i\right) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H) \end{pmatrix}\,, \tag{9.58}$$

so that the original two complex scalar fields φ^+ and φ^0 are replaced by four real scalar fields H, ξ_1, ξ_2, and ξ_3. All these fields have zero vacuum expectation values:

$$\langle 0\,|\,\xi_i\,|\,0\rangle = \langle 0\,|\,H\,|\,0\rangle = 0\,. \tag{9.59}$$

We will now reformulate the model in the *unitary gauge* where the three would-be Goldstone bosons ξ_i are transformed away, bringing out in a particularly transparent way the spectra and interactions of the remaining physical particles. First, apply the unitary transformation

$$\mathcal{S} = \exp\left(-\frac{\mathrm{i}}{v}\sum \xi_i T_i\right) \tag{9.60}$$

on all fields, resulting in the transformed fields

$$\begin{aligned}
\boldsymbol{\phi}' &= \mathcal{S}\boldsymbol{\phi} = \tfrac{1}{\sqrt{2}}[v + H(x)]\chi\,, \quad \text{with} \quad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,; \\
\psi'_{\mathrm{L}} &= \mathcal{S}\psi_{\mathrm{L}}\,; \qquad \psi'_{\mathrm{R}} = \psi_{\mathrm{R}}\,; \\
B'_\mu &= B_\mu\,; \\
\boldsymbol{A}'_\mu &= \mathcal{S}\boldsymbol{A}_\mu\,\mathcal{S}^\dagger + \frac{\mathrm{i}}{g}\,(\partial_\mu \mathcal{S})\,\mathcal{S}^\dagger\,.
\end{aligned} \tag{9.61}$$

The Lagrangian of the model is of course invariant to these transformations. In terms of the new fields, its different parts become

$$\mathcal{L}_{\rm s} = \left(D'_\mu \boldsymbol{\phi}'\right)^\dagger \left(D'^\mu \boldsymbol{\phi}'\right) - \mu^2 \boldsymbol{\phi}'^\dagger \boldsymbol{\phi}' - \lambda \left(\boldsymbol{\phi}'^\dagger \boldsymbol{\phi}'\right)^2 ; \tag{9.62}$$

$$\mathcal{L}_{\rm G} = -\frac{1}{4} W'^i_{\mu\nu} W'^{\mu\nu}_i - \frac{1}{4} B'_{\mu\nu} B'^{\mu\nu} ; \tag{9.63}$$

$$\mathcal{L}_\ell = \quad \overline{\psi}'_{\rm L}\, {\rm i}\gamma^\mu D'^{\rm L}_\mu\, \psi'_{\rm L} + \overline{\psi}'_{\rm R}\, {\rm i}\gamma^\mu D'^{\rm R}_\mu\, \psi'_{\rm R} ; \tag{9.64}$$

$$\mathcal{L}_{\ell{\rm Y}} = -C_{\rm e} \left[\overline{\psi}'_{\rm R} (\boldsymbol{\phi}'^\dagger \psi'_{\rm L}) + (\overline{\psi}'_{\rm L} \boldsymbol{\phi}') \psi'_{\rm R} \right] . \tag{9.65}$$

Each of these parts is now examined in turn. For brevity we will drop the prime accents on the field symbols, identifying for example ψ' with ψ.

Scalar Fields. The main role of the scalar fields is to generate masses for the gauge bosons and the electron. The gauge field matrix may be written explicitly as

$$\begin{aligned} \boldsymbol{A}_\mu = \tfrac{1}{2}\, \tau_i A_{i\mu} &= \tfrac{1}{2} \left(\tau_1 A_{1\mu} + \tau_2 A_{2\mu} + \tau_3 A_{3\mu}\right) \\ &= \tfrac{1}{\sqrt{2}} (\tau_+\, W_\mu + \tau_-\, W^\dagger_\mu) + \tfrac{1}{2}\, \tau_3\, A_{3\mu} , \end{aligned}$$

with the definitions $\tau_\pm = \frac{1}{2}\,(\tau_1 \pm {\rm i}\tau_2)$ and $W_\mu = \frac{1}{\sqrt{2}}(A_{1\mu} - {\rm i}A_{2\mu})$. Then the covariant derivative of the scalar is

$$\begin{aligned} D_\mu \boldsymbol{\phi} &= \left(\partial_\mu + {\rm i}g\, \boldsymbol{A}_\mu + {\rm i}g'\, B_\mu\, \frac{Y_{\rm H}}{2}\right) \frac{v+H}{\sqrt{2}}\, \chi \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} {\rm i}g\, W_\mu\, (v+H) \\ \partial_\mu H - \frac{1}{2} {\rm i}(g\, A_{3\mu} - g'\, B_\mu)(v+H) \end{pmatrix} . \end{aligned} \tag{9.66}$$

The vector meson masses are found in the ‘kinetic term’

$$\begin{aligned} &(D_\mu \boldsymbol{\phi})^\dagger (D^\mu \boldsymbol{\phi}) \\ &\quad = \tfrac{1}{4}\, g^2\, (v+H)^2\, W^\dagger_\mu W^\mu + \tfrac{1}{2}\ \left[\partial_\mu H\, \partial^\mu H + \tfrac{1}{4}\, (v+H)^2\, (g\, A_{3\mu} - g'\, B_\mu)^2\right] \\ &\quad = \tfrac{1}{4} g^2 v^2\, W^\dagger_\mu W^\mu + \tfrac{1}{8} v^2\, (g A_{3\mu} - g' B_\mu)^2 + \tfrac{1}{2}\, \partial_\mu H\, \partial^\mu H \\ &\qquad + \tfrac{1}{4} (2vH + H^2)\ \left[g^2\, W^\dagger_\mu W^\mu + \tfrac{1}{2}\, (g\, A_{3\mu} - g'\, B_\mu)^2\right] . \end{aligned} \tag{9.67}$$

In general, the expected mass term for a complex vector field is of the form $M^2_{\rm W}\, W^\dagger_\mu W^\mu$. So that the mass for the charged vector mesons can be read off from (67):

$$M_{\rm W} = \tfrac{1}{2}\, g\, v . \tag{9.68}$$

On the other hand, the quadratic terms in the neutral fields,

$$\tfrac{1}{8} v^2\, (g\, A_{3\mu} - g'\, B_\mu)^2 , \tag{9.69}$$

contain the nondiagonal matrix

$$\frac{1}{8}v^2\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} .$$

To diagonalize it, we introduce the orthogonal combinations that give the mass eigenstates for the two neutral fields

$$A_\mu = \sin\theta_\mathrm{W}\ A_{3\mu} + \cos\theta_\mathrm{W}\, B_\mu\,, \tag{9.70}$$

$$Z_\mu = \cos\theta_\mathrm{W}\, A_{3\mu} - \sin\theta_\mathrm{W}\, B_\mu\,; \tag{9.71}$$

or, inversely,

$$A_{3\mu} = \sin\theta_\mathrm{W}\, A_\mu + \cos\theta_\mathrm{W}\, Z_\mu\,, \tag{9.72}$$

$$B_\mu = \cos\theta_\mathrm{W}\, A_\mu - \sin\theta_\mathrm{W}\, Z_\mu\,; \tag{9.73}$$

where θ_W is a mixing angle (called the Weinberg angle) yet to be determined. Substituting these expressions into (69) yields

$$\begin{aligned} &\tfrac{1}{8}v^2\,(g\,A_{3\mu} - g'\,B_\mu)^2 \\ &\quad = \tfrac{1}{8}v^2\left[A_\mu^2\,(g\sin\theta_\mathrm{W} - g'\cos\theta_\mathrm{W})^2 + Z_\mu^2\,(g\cos\theta_\mathrm{W} + g'\sin\theta_\mathrm{W})^2\right. \\ &\qquad \left. + \, 2A_\mu\, Z^\mu\,(g\sin\theta_\mathrm{W} - g'\cos\theta_\mathrm{W})(g\cos\theta_\mathrm{W} + g'\sin\theta_\mathrm{W})\,\right] . \end{aligned} \tag{9.74}$$

As we have seen in (57), $\mathrm{U_Q}(1)$ is unbroken and the associated gauge boson (the photon) remains massless. If we let the corresponding field be A_μ, the diagonalization of (74) yields the condition

$$g\sin\theta_\mathrm{W} = g'\cos\theta_\mathrm{W}\,. \tag{9.75}$$

Thus, the mixing angle θ_W gives a measure of the relative strength of the SU(2) and U(1) group factors; it may be calculated from the relations

$$\cos\theta_\mathrm{W} = \frac{g}{\sqrt{g^2+g'^2}}\,; \qquad \sin\theta_\mathrm{W} = \frac{g'}{\sqrt{g^2+g'^2}}\,. \tag{9.76}$$

Since these functions will recur again and again in the following, a simplified notation is called for:

$$c_\mathrm{W} \equiv \cos\theta_\mathrm{W}\,, \qquad s_\mathrm{W} \equiv \sin\theta_\mathrm{W}\,. \tag{9.77}$$

The quadratic form (74) must reduce to the expected mass term for a neutral vector field, $\frac{1}{2}\,M_\mathrm{Z}^2 Z_\mu Z^\mu$, so that the mass of the field Z_μ is

$$\begin{aligned} M_\mathrm{Z} &= \tfrac{1}{2}v\,(g c_\mathrm{W} + g' s_\mathrm{W}) = \tfrac{1}{2}v\sqrt{g^2+g'^2} \\ &= \frac{gv}{2\,c_\mathrm{W}}\,. \end{aligned} \tag{9.78}$$

Note that the masses of the two weak gauge fields satisfy the identity

$$M_{\rm W} = c_{\rm W}\, M_{\rm Z}\,. \tag{9.79}$$

On the other hand, the potential becomes after symmetry breaking

$$\begin{aligned} V(\boldsymbol{\phi}) &= \tfrac{1}{2}\,\mu^2 (v+H)^2\,(\chi^\dagger\chi) + \tfrac{1}{4}\lambda\,(v+H)^4\,(\chi^\dagger\chi)^2 \\ &= \tfrac{1}{4}\mu^2 v^2 - \mu^2 H^2 + \lambda\,(vH^3 + \tfrac{1}{4}H^4)\,, \end{aligned} \tag{9.80}$$

where $v^2 = -\mu^2/\lambda$, from which the mass of the surviving scalar can be identified:

$$M_{\rm H}^2 = -2\mu^2\,. \tag{9.81}$$

The Lagrangian $\mathcal{L}_{\rm s}$ thus becomes in the unitary gauge

$$\begin{aligned} \mathcal{L}_{\rm s} =& (D_\mu\boldsymbol{\phi})^\dagger (D_\mu\boldsymbol{\phi}) - V(\boldsymbol{\phi}) \\ =& \frac{1}{2}(\partial_\mu H \partial^\mu H - M_{\rm H}^2 H^2) - \frac{gM_{\rm H}^2}{4\,M_{\rm W}}\,H^3 - \frac{g^2 M_{\rm H}^2}{32\,M_{\rm W}^2}\,H^4 \\ &+ gM_{\rm W}\left(H + \frac{g}{4M_{\rm W}}H^2\right) W_\mu^\dagger W^\mu + M_{\rm W}^2 W_\mu^\dagger W^\mu \\ &+ \frac{1}{2}\,\frac{gM_{\rm Z}}{c_{\rm W}}\left(H + \frac{g}{4c_{\rm W}\,M_{\rm Z}}H^2\right) Z_\mu Z^\mu + \frac{1}{2}\,M_{\rm Z}^2\,Z_\mu Z^\mu\,. \end{aligned} \tag{9.82}$$

Here we have replaced the original parameters by the particle masses; in particular

$$v = \frac{2M_{\rm W}}{g} = \frac{2M_{\rm Z}}{\sqrt{g^2+g'^2}}\,; \qquad \lambda = -\frac{\mu^2}{v^2} = \frac{M_{\rm H}^2}{2\,v^2} = \frac{g^2 M_{\rm H}^2}{8\,M_{\rm W}^2}\,. \tag{9.83}$$

The field H, being electrically neutral, is not coupled to the electromagnetic field, but $\mathcal{L}_{\rm s}$ is nonetheless $U_{\rm Q}(1)$-invariant.

Gauge Fields. The Lagrangian for the gauge fields $\mathcal{L}_{\rm G}$ will be split into free-field and interacting-field parts:

$$\mathcal{L}_{\rm G} = \mathcal{L}_{\rm G}^0 + \mathcal{L}_{\rm G}^1 + \mathcal{L}_{\rm G}^2\,; \tag{9.84}$$

$$\mathcal{L}_{\rm G}^0 = -\frac{1}{4}\,A_{\mu\nu}^i A_i^{\mu\nu} - \frac{1}{4}\,B_{\mu\nu}B^{\mu\nu}\,, \tag{9.85}$$

$$\mathcal{L}_{\rm G}^1 = \frac{1}{2}\,g\epsilon_{ijk}\,A_\mu^j A_\nu^k\,A_i^{\mu\nu}\,, \tag{9.86}$$

$$\mathcal{L}_{\rm G}^2 = -\frac{1}{4}\,g^2 \epsilon_{ijk}\epsilon_{i\ell m}\,A_\mu^j A_\nu^k A_\ell^\mu A_m^\nu\,. \tag{9.87}$$

The two interacting-field terms are characteristic of non-Abelian theories, with the structure constants of SU(2) algebra. Here the customary symbols for field strengths have been used:

$$A^i_{\mu\nu} \equiv \partial_\mu A^i_\nu - \partial_\nu A^i_\mu\,, \qquad B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{9.88}$$

Our main task is to re-express $\mathcal{L}_{\rm G}$ in terms of the mass eigenstates A_μ, W_μ, and Z_μ. It is convenient for this purpose to introduce their respective field strengths:

$$\begin{aligned} &A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,\\ &W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu\,, \qquad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu\,. \end{aligned} \tag{9.89}$$

Noting that

$$\begin{aligned} &A^1_{\mu\nu}A^{\mu\nu}_1 + A^2_{\mu\nu}A^{\mu\nu}_2 = 2\,W^\dagger_{\mu\nu}W^{\mu\nu}\,,\\ &A^3_{\mu\nu}A^{\mu\nu}_3 + B_{\mu\nu}B^{\mu\nu} = A_{\mu\nu}A^{\mu\nu} + Z_{\mu\nu}Z^{\mu\nu}\,, \end{aligned}$$

we readily get the *kinetic part*

$$\mathcal{L}^0_{\rm G} = -\frac{1}{2}\,W^\dagger_{\mu\nu}W^{\mu\nu} - \frac{1}{4}\,Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}\,A_{\mu\nu}A^{\mu\nu}\,. \tag{9.90}$$

The *three-field coupling* $\mathcal{L}^1_{\rm G}$ contains two factors, the first of which, $\epsilon_{ijk}\,A^j_\mu A^k_\nu$, gives the terms

$$\begin{aligned} \epsilon_{1jk}A_{j\mu}A_{k\nu} &= A_{2\mu}A_{3\nu} - A_{2\nu}A_{3\mu}\\ &= -\tfrac{\rm i}{\sqrt{2}}\,\big[(W^\dagger_\mu - W_\mu)(s_{\rm W}A_\nu + c_{\rm W}Z_\nu) - (W^\dagger_\nu - W_\nu)(s_{\rm W}A_\mu + c_{\rm W}Z_\mu)\big]\,,\\ \epsilon_{2jk}A_{j\mu}A_{k\nu} &= A_{3\mu}A_{1\nu} - A_{3\nu}A_{1\mu}\\ &= \tfrac{1}{\sqrt{2}}\,\big[-(W^\dagger_\mu + W_\mu)(s_{\rm W}A_\nu + c_{\rm W}Z_\nu) + (W^\dagger_\nu + W_\nu)(s_{\rm W}A_\mu + c_{\rm W}Z_\mu)\big]\,,\\ \epsilon_{3jk}A_{j\mu}A_{k\nu} &= A_{1\mu}A_{2\nu} - A_{1\nu}A_{2\mu} = {\rm i}(W^\dagger_\mu W_\nu - W^\dagger_\nu W_\mu)\,; \end{aligned}$$

and the second, $A^{\mu\nu}_i$, the following:

$$\begin{aligned} &A^1_{\mu\nu} = \tfrac{1}{\sqrt{2}}(W^\dagger_{\mu\nu} + W_{\mu\nu})\,,\\ &A^2_{\mu\nu} = -\tfrac{\rm i}{\sqrt{2}}(W^\dagger_{\mu\nu} - W_{\mu\nu})\,,\\ &A^3_{\mu\nu} = c_{\rm W}Z_{\mu\nu} + s_{\rm W}A_{\mu\nu}\,. \end{aligned}$$

Together, they lead to

$$\begin{aligned} \mathcal{L}^1_{\rm G} =& \frac{1}{2}\,g\,\epsilon_{ijk}\,A_{j\mu}A_{k\nu}A^{\mu\nu}_i = {\rm i}gW^{\mu\dagger}W^\nu(s_{\rm W}A_{\mu\nu} + c_{\rm W}Z_{\mu\nu})\\ &+ {\rm i}g\,(W^\mu W^\dagger_{\mu\nu} - W^{\mu\dagger}W_{\mu\nu})\,(s_{\rm W}A^\nu + c_{\rm W}Z^\nu)\,. \end{aligned} \tag{9.91}$$

As for the *four-field coupling* $\mathcal{L}^2_{\mathrm{G}}$, it suffices to note that

$$\begin{aligned} A_{j\mu}A_{j\nu} &= A_{1\mu}A_{1\nu} + A_{2\mu}A_{2\nu} + A_{3\mu}A_{3\nu} \\ &= W^\dagger_\mu W_\nu + W_\mu W^\dagger_\nu + (s_{\mathrm{W}} A_\mu + c_{\mathrm{W}} Z_\mu)\,(s_{\mathrm{W}} A_\nu + c_{\mathrm{W}} Z_\nu)\,. \end{aligned} \tag{9.92}$$

Then $\mathcal{L}^2_{\mathrm{G}}$ can be rewritten in the desired form

$$\begin{aligned} \mathcal{L}^2_{\mathrm{G}} &= -\,\frac{1}{4}\,g^2 \epsilon_{ijk}\epsilon_{i\ell m}\, A^j_\mu A^k_\nu A^\mu_\ell A^\nu_m \\ &= -\,\frac{1}{4}\,g^2 \left[(A_{j\mu}A^\mu_j)(A_{k\nu}A^\nu_k) - (A_{j\mu}A^\nu_j)(A_{k\nu}A^\mu_k)\right] \\ &= -\,\frac{1}{2}\,g^2 \left(W^\dagger_\mu W^\mu\, W^\dagger_\nu W^\nu - W^{\mu\dagger} W^\dagger_\mu\, W^\nu W_\nu\right) \\ &\quad - g^2\, W^\dagger_\mu W^\mu (s^2_{\mathrm{W}} A_\nu A^\nu + c^2_{\mathrm{W}} Z_\nu Z^\nu + 2\, s_{\mathrm{W}}\, c_{\mathrm{W}} A_\nu Z^\nu) \\ &\quad + g^2 W^\dagger_\mu W_\nu \left[s^2_{\mathrm{W}} A^\mu A^\nu + c^2_{\mathrm{W}} Z^\mu Z^\nu + s_{\mathrm{W}} c_{\mathrm{W}} (A^\mu Z^\nu + A^\nu Z^\mu)\right]. \end{aligned} \tag{9.93}$$

The terms that depend only on the neutral fields (e.g. $A^2 Z^2$ or Z^4) have canceled out, so that all the remaining terms in $\mathcal{L}^2_{\mathrm{G}}$ involve charged bosons. These must be coupled to A_μ in a $\mathrm{U_Q}(1)$-gauge-invariant way. To check that this is really so, we sum the terms

$$\begin{array}{rll} -\frac{1}{2}\, W^\dagger_{\mu\nu} W^{\mu\nu} & \text{from} & \mathcal{L}^0_{\mathrm{G}}\,, \\ \mathrm{i} g s_{\mathrm{W}}\, A^\nu (W^\dagger_{\mu\nu} W^\mu - W^{\mu\dagger} W_{\mu\nu}) & \text{from} & \mathcal{L}^1_{\mathrm{G}}\,, \\ -g^2 s^2_{\mathrm{W}}\, (W^\dagger_\mu W^\mu A_\nu A^\nu - W^\dagger_\mu W_\nu\, A^\mu A^\nu) & \text{from} & \mathcal{L}^2_{\mathrm{G}} \end{array}$$

into a single expression

$$\begin{aligned} -\tfrac{1}{2} &\left[W^\dagger_{\mu\nu} W^{\mu\nu} - 2\,\mathrm{i} g s_{\mathrm{W}}\, A^\nu (W^\dagger_{\mu\nu} W^\mu - W^{\mu\dagger} W_{\mu\nu}) \right. \\ &\qquad \left. + \,2\,(g s_{\mathrm{W}})^2\, (W^\dagger_\mu W^\mu A_\nu A^\nu - W^\dagger_\mu A^\mu W_\nu A^\nu)\right] \\ &= -\tfrac{1}{2}\,(D_\mu W_\nu - D_\nu W_\mu)^\dagger (D^\mu W^\nu - D^\nu W^\mu)\,. \end{aligned} \tag{9.94}$$

Here the covariant derivative of W_μ is defined as

$$D_\mu W_\nu = (\partial_\mu + \mathrm{i}\, g s_{\mathrm{W}}\, A_\mu) W_\nu\,. \tag{9.95}$$

It now becomes clear that W_μ is a field carrying a positive electrical charge $g s_{\mathrm{W}}$ interacting with the electromagnetic field via an expected $\mathrm{U_Q}(1)$-invariant coupling. Not surprisingly, both A_μ and Z_μ are neutral.

The Lepton Sector. We now consider $\mathcal{L}_\ell$ and $\mathcal{L}_{\ell\mathrm{Y}}$. We anticipate that after symmetry breaking the electron becomes massive from its coupling to the scalars. It is indeed the case because with

$$\bar{\psi}_{\mathrm{L}} \boldsymbol{\phi} = \begin{pmatrix} \bar{\nu}_{\mathrm{L}} & \bar{e}_{\mathrm{L}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{v + H}{\sqrt{2}} = \frac{v + H}{\sqrt{2}}\, \bar{e}_{\mathrm{L}}\,, \tag{9.96}$$

the Yukawa couplings become

$$\begin{aligned}\mathcal{L}_{\ell\mathrm{Y}} &= -C_\mathrm{e}\,\left[\overline{\psi}_\mathrm{R}\,(\boldsymbol{\phi}^\dagger\psi_\mathrm{L}) + (\overline{\psi}_\mathrm{L}\boldsymbol{\phi})\psi_\mathrm{R}\right] \\ &= -\frac{C_\mathrm{e}}{\sqrt{2}}(v+H)(\bar{e}_\mathrm{R}e_\mathrm{L} + \bar{e}_\mathrm{L}e_\mathrm{R}) = -\frac{C_\mathrm{e}}{\sqrt{2}}(v+H)\,\overline{e}e\,. \end{aligned} \tag{9.97}$$

The term quadratic in the electron field should be recognized as the Dirac mass term for the electron, $-m_\mathrm{e}\,\overline{e}e$, with mass

$$m_\mathrm{e} = \frac{C_\mathrm{e}v}{\sqrt{2}}\,, \tag{9.98}$$

which may be inverted to give the Yukawa coupling strength

$$C_\mathrm{e} = \frac{\sqrt{2}\,m_\mathrm{e}}{v} = \frac{gm_\mathrm{e}}{\sqrt{2}\,M_\mathrm{W}}\,. \tag{9.99}$$

The scalar–electron coupling now reduces to

$$\mathcal{L}_{\ell\mathrm{Y}} = -m_\mathrm{e}\,\overline{e}e - \frac{gm_\mathrm{e}}{2\,M_\mathrm{W}}\,H\overline{e}e\,. \tag{9.100}$$

Finally, we come to the gauge-invariant part of the electron and its neutrino

$$\mathcal{L}_\ell = \overline{\psi}_\mathrm{L}\,\mathrm{i}\gamma^\mu D^\mathrm{L}_\mu\,\psi_\mathrm{L} + \overline{\psi}_\mathrm{R}\,\mathrm{i}\gamma^\mu D^\mathrm{R}_\mu\,\psi_\mathrm{R}\,. \tag{9.101}$$

In more detail, one has for the first term on the right-hand side

$$\begin{aligned}\overline{\psi}_\mathrm{L}\mathrm{i}\gamma^\mu D^\mathrm{L}_\mu\,\psi_\mathrm{L} &= \overline{\psi}_\mathrm{L}\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}g\boldsymbol{A}_\mu - \tfrac{\mathrm{i}}{2}g'B_\mu)\psi_\mathrm{L} \\ &= \overline{\psi}_\mathrm{L}\mathrm{i}\gamma^\mu\left[\partial_\mu + \tfrac{\mathrm{i}}{\sqrt{2}}g(W_\mu\tau_+ + W^\dagger_\mu\tau_-) + \tfrac{\mathrm{i}}{2}(gA_{3\mu}\tau_3 - g'B_\mu)\right]\psi_\mathrm{L}\,;\end{aligned}$$

and for the second term

$$\begin{aligned}\overline{\psi}_\mathrm{R}\,\mathrm{i}\gamma^\mu D^\mathrm{R}_\mu\,\psi_\mathrm{R} &= \overline{\psi}_\mathrm{R}\,\mathrm{i}\gamma^\mu\left(\partial_\mu + \mathrm{i}g'\frac{Y_\mathrm{R}}{2}B_\mu\right)\psi_\mathrm{R} \\ &= \overline{e}_\mathrm{R}\mathrm{i}\gamma^\mu\partial_\mu e_\mathrm{R} + g'\,\overline{e}_\mathrm{R}\gamma^\mu e_\mathrm{R}\,B_\mu\,.\end{aligned}$$

They contain the expected kinetic terms for the two leptons

$$\overline{\psi}_\mathrm{L}\mathrm{i}\gamma^\mu\partial_\mu\psi_\mathrm{L} + \overline{\psi}_\mathrm{R}\mathrm{i}\gamma^\mu\partial_\mu\psi_\mathrm{R} = \overline{\nu}_\mathrm{L}\,\mathrm{i}\gamma^\mu\,\partial_\mu\nu_\mathrm{L} + \overline{e}\,\mathrm{i}\gamma^\mu\,\partial_\mu e \tag{9.102}$$

as well as their various interactions with the gauge bosons. First, we have the charge-changing couplings

$$\begin{aligned}\mathcal{L}^\ell_\mathrm{cc} &= -\frac{g}{\sqrt{2}}\,\left[(\overline{\psi}_\mathrm{L}\gamma^\mu\tau_+\psi_\mathrm{L})\,W_\mu + (\overline{\psi}_\mathrm{L}\gamma^\mu\tau_-\psi_\mathrm{L})\,W^\dagger_\mu\right] \\ &= -\frac{g}{\sqrt{2}}\,(J^{\mu\dagger}W_\mu + J^\mu W^\dagger_\mu)\,,\end{aligned} \tag{9.103}$$

where the charged currents are defined as

$$J_\mu = j^1_\mu - \mathrm{i} j^2_\mu = \overline{\psi}_\mathrm{L}\gamma_\mu\tau_-\psi_\mathrm{L} = \overline{e}_\mathrm{L}\gamma_\mu\nu_\mathrm{L}\,; \tag{9.104}$$

$$J^\dagger_\mu = j^1_\mu + \mathrm{i} j^2_\mu = \overline{\psi}_\mathrm{L}\gamma_\mu\tau_+\psi_\mathrm{L} = \overline{\nu}_\mathrm{L}\gamma_\mu e_\mathrm{L}\,. \tag{9.105}$$

Next, we reshape the remaining terms which describe the couplings to the neutral gauge fields in a similar form:

$$\begin{aligned}\mathcal{L}^\ell_\mathrm{nc} &= -g\,A_{3\mu}\left(\overline{\psi}_\mathrm{L}\gamma^\mu\frac{\tau_3}{2}\psi_\mathrm{L}\right) - \frac{1}{2}\,g'\,B_\mu\left(-\overline{\psi}_\mathrm{L}\gamma^\mu\psi_\mathrm{L} - 2\overline{\psi}_\mathrm{R}\gamma^\mu\psi_\mathrm{R}\right)\\ &= -g\,j^3_\mu A^\mu_3 - \frac{1}{2}\,g'\,j^\mathrm{Y}_\mu B^\mu\,,\end{aligned} \tag{9.106}$$

where we have used (17) and (37). Re-expressing the gauge group eigenstates A^3_μ and B_μ in terms of the mass eigenstates A_μ and Z_μ, the couplings take the form

$$\mathcal{L}^\ell_\mathrm{nc} = -\left(g s_\mathrm{W}\,j^3_\mu + \frac{1}{2}\,g' c_\mathrm{W}\,j^\mathrm{Y}_\mu\right)A^\mu - \left(g c_\mathrm{W}\,j^3_\mu - \frac{1}{2}\,g' s_\mathrm{W}\,j^\mathrm{Y}_\mu\right)Z^\mu\,. \tag{9.107}$$

As the various neutral currents are related through $j^\mathrm{em}_\mu = j^3_\mu + \frac{1}{2}\,j^\mathrm{Y}_\mu$, the vector current to which A_μ is coupled may be written as

$$\begin{aligned}g\,s_\mathrm{W}\,j^3_\mu + \tfrac{1}{2}\,g'\,c_\mathrm{W}\,j^\mathrm{Y}_\mu &= g'\,c_\mathrm{W}\,j^\mathrm{em}_\mu + (g\,s_\mathrm{W} - g'\,c_\mathrm{W})\,j^3_\mu\\ &= g s_\mathrm{W}\,j^\mathrm{em}_\mu + \tfrac{1}{2}\,(g' c_\mathrm{W} - g s_\mathrm{W})\,j^\mathrm{Y}_\mu\,.\end{aligned}$$

The second terms on the right-hand sides of the last two equations vanish by (75), and the first terms should be recognized as the electromagnetic current, $e j^\mathrm{em}_\mu$. From this follows a relation between the coupling constants associated with the original symmetries and the residual symmetry:

$$e = g s_\mathrm{W} = g' c_\mathrm{W} \qquad \text{or} \qquad \frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}\,. \tag{9.108}$$

On the other hand, the vector field coupled to Z^μ is

$$g c_\mathrm{W}\,j^3_\mu - \tfrac{1}{2}\,g' s_\mathrm{W}\,j^\mathrm{Y}_\mu = g\,c^{-1}_\mathrm{W}\,(j^3_\mu - s^2_\mathrm{W}\,j^\mathrm{em}_\mu) \equiv g\,c^{-1}_\mathrm{W}\,j^\mathrm{Z}_\mu\,. \tag{9.109}$$

Here we have introduced the weak neutral current

$$\begin{aligned}j^\mathrm{Z}_\mu &= j^3_\mu - s^2_\mathrm{W}\,j^\mathrm{em}_\mu\\ &= \overline{\psi}_\mathrm{L}\gamma_\mu T_3\psi_\mathrm{L} - s^2_\mathrm{W}\left[Q_\mathrm{e}(\overline{e}_\mathrm{L}\gamma_\mu e_\mathrm{L} + \overline{e}_\mathrm{R}\gamma_\mu e_\mathrm{R})\right]\\ &= \overline{\psi}_\mathrm{L}\gamma_\mu Z_\mathrm{L}\psi_\mathrm{L} + \overline{\psi}_\mathrm{R}\gamma_\mu Z_\mathrm{R}\psi_\mathrm{R}\,,\end{aligned} \tag{9.110}$$

where, in analogy with the electric charges, we have defined the ‘weak charges’

$$Z_\mathrm{L} = T_{3\mathrm{L}} - Q s^2_\mathrm{W}\,, \quad Z_\mathrm{R} = -Q s^2_\mathrm{W}\,.$$

The weak neutral current is a novel feature of the unified model. It differs from the charged current in many ways – by having a characteristic charge $T_3 - Qs_{\mathrm{W}}^2$, by being diagonal in flavor, and by containing both left and right chiral components of the electron. It also differs from the electromagnetic current in that it involves both neutral and charged lepton fields.

After these transformations, the complete neutral current couplings reduce to a compact expression

$$\mathcal{L}_{\mathrm{nc}}^{\ell} = -e\, j_\mu^{\mathrm{em}} A^\mu - \frac{g}{c_{\mathrm{W}}}\, j_\mu^{\mathrm{Z}} Z^\mu \,. \tag{9.111}$$

The e.m. coupling $-e\, j_\mu^{\mathrm{em}} A^\mu$, together with the kinetic terms (102),

$$\overline{\nu}_{\mathrm{L}} \mathrm{i}\gamma^\mu \partial_\mu \nu_{\mathrm{L}} + \overline{e}\mathrm{i}\gamma^\mu \partial_\mu e + e\,\overline{e}\gamma_\mu e\, A^\mu = \overline{\nu}_{\mathrm{L}} \mathrm{i}\gamma^\mu \partial_\mu \nu_{\mathrm{L}} + \overline{e}\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}e\, A_\mu)e\,,$$

makes explicit the familiar minimal coupling and the $\mathrm{U_Q}(1)$ gauge invariance of the lepton Lagrangian.

9.2.4 Feynman Rules for One-Lepton Family

For convenience we now gather together the results of this section, rewriting them in a slightly more logical arrangement. The $\mathrm{W}^\pm$ and Z mass terms from $\mathcal{L}_{\mathrm{s}}$ are joined with $\mathcal{L}_{\mathrm{G}}$ to give

$$\begin{aligned}
\mathcal{L}_{\mathrm{G}}'^{0} =& -\frac{1}{2}\, W_{\mu\nu}^\dagger W^{\mu\nu} + M_{\mathrm{W}}^2 W_\mu^\dagger W^\mu \\
& - \frac{1}{4}\, Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{2}\, M_{\mathrm{Z}}^2 Z_\mu Z^\mu - \frac{1}{4}\, A_{\mu\nu} A^{\mu\nu} ;
\end{aligned} \tag{9.112}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{G}}^{1} =& \mathrm{i}g s_{\mathrm{W}}\, \left(W^{\mu\dagger} W^\nu A_{\mu\nu} + W_{\mu\nu}^\dagger W^\mu A^\nu - W_{\mu\nu} W^{\mu\dagger} A^\nu \right) \\
& + \mathrm{i}g c_{\mathrm{W}}\, \left(W^{\mu\dagger} W^\nu Z_{\mu\nu} + W_{\mu\nu}^\dagger W^\mu Z^\nu - W_{\mu\nu} W^{\mu\dagger} Z^\nu \right)\, ;
\end{aligned} \tag{9.113}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{G}}^{2} =& -(g s_{\mathrm{W}})^2 \,(W_\mu^\dagger W^\mu\, A_\nu A^\nu - W_\mu^\dagger W_\nu\, A^\mu A^\nu) \\
& - (g c_{\mathrm{W}})^2 (W_\mu^\dagger W^\mu\, Z_\nu Z^\nu - W_\mu^\dagger W_\nu\, Z^\mu Z^\nu) \\
& - (g s_{\mathrm{W}})(g c_{\mathrm{W}})\,[2\, W_\mu^\dagger W^\mu\, A_\nu Z^\nu - W_\mu^\dagger W_\nu\, (A^\mu Z^\nu + A^\nu Z^\mu)] \\
& + \frac{1}{2}\, g^2\, (W_\mu^\dagger W^{\mu\dagger} W_\nu W^\nu - W_\mu^\dagger W^\mu W_\nu^\dagger W^\nu)\,.
\end{aligned} \tag{9.114}$$

The original Lagrangian for scalars, minus the $\mathrm{W}^\pm$ and Z mass terms, represents the Higgs field and its interactions with the vector bosons:

$$\begin{aligned}
\mathcal{L}_{\mathrm{s}}' =& \frac{1}{2}(\partial_\mu H \partial^\mu H - M_{\mathrm{H}}^2 H^2) - \frac{1}{3!}\, \frac{3\, g M_{\mathrm{H}}^2}{2\, M_{\mathrm{W}}}\, H^3 - \frac{1}{4!}\, \frac{3\, g^2 M_{\mathrm{H}}^2}{4\, M_{\mathrm{W}}^2}\, H^4 \\
& + g M_{\mathrm{W}} \left(H + \frac{g}{2\, M_{\mathrm{W}}}\, \frac{1}{2} H^2 \right) W_\mu^\dagger W^\mu \\
& + \frac{1}{2}\, \frac{g M_{\mathrm{Z}}}{c_{\mathrm{W}}} \left(H + \frac{g}{2\, c_{\mathrm{W}} M_{\mathrm{Z}}}\, \frac{1}{2} H^2 \right) Z_\mu Z^\mu \,.
\end{aligned} \tag{9.115}$$

The Yukawa coupling, minus the electron mass term, reduces to the electron–Higgs coupling

$$\mathcal{L}'_{\ell \mathrm{H}} = -\frac{g m_\mathrm{e}}{2\, M_\mathrm{W}}\, \overline{e} e\, H\,. \tag{9.116}$$

Finally, the lepton sector is described by $\mathcal{L}_\ell$ augmented by the electron mass term from the original $\mathcal{L}_\mathrm{s}$,

$$\mathcal{L}'^{\ell}_{\mathrm{kin}} = \quad \overline{\nu}_\mathrm{L} \mathrm{i}\gamma^\mu \partial_\mu \nu_\mathrm{L} + \overline{e}\left[\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu) - m_\mathrm{e}\right] e\,; \tag{9.117}$$

$$\mathcal{L}^{\ell}_{\mathrm{cc}} = -\frac{g}{2\sqrt{2}} \left[\overline{e}\gamma^\mu(1-\gamma_5)\nu\, W^\dagger_\mu + \overline{\nu}\gamma^\mu(1-\gamma_5) e\, W_\mu\right]\,; \tag{9.118}$$

$$\mathcal{L}^{\ell}_{\mathrm{nc}} = -\frac{g}{4c_\mathrm{W}}\, \overline{\nu}\gamma^\mu(1-\gamma_5)\nu\, Z_\mu - \frac{g}{4c_\mathrm{W}} \overline{e}\gamma^\mu \left[(-1+4s^2_\mathrm{W}) + \gamma_5\right] e\, Z_\mu\,. \tag{9.119}$$

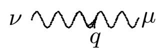
The sum of (112)–(119) gives the Lagrangian of the gauge model of the electroweak interaction for the electron-type lepton family. This model contains five independent parameters. Before symmetry breaking, they are the SU(2) coupling g, the $\mathrm{U_Y}(1)$ coupling g', the scalar potential parameters λ and μ^2, and the Yukawa coupling C_e. After symmetry breaking, they may be equivalently replaced by the absolute value of the electron charge e, the Weinberg mixing angle θ_W, the electron mass m_e, the Higgs mass M_H, and the mass of the charged boson M_W. The mass of the neutral boson is not independent, being $M_\mathrm{Z} = M_\mathrm{W}/\cos\theta_\mathrm{W}$. The two sets of parameters are related through

$$\begin{aligned}
\tan\theta_\mathrm{W} &= \frac{g'}{g}\,, \\
e &= g\sin\theta_\mathrm{W}\,, \\
m_\mathrm{e} &= \frac{1}{\sqrt{2}} C_\mathrm{e} v = \frac{1}{\sqrt{2}} C_\mathrm{e} \sqrt{-\mu^2/\lambda}\,, \\
M_\mathrm{W} &= \frac{1}{2}\, g v = \frac{1}{2}\, g\sqrt{-\mu^2/\lambda}\,, \\
M_\mathrm{H} &= \sqrt{-2\mu^2}\,.
\end{aligned} \tag{9.120}$$

The fact that interactions of all gauge fields are determined by the electric charge and one free parameter (the Weinberg angle) is noteworthy. It is proof that the standard model is a unified theory of the weak and electromagnetic interactions, but also that the unification is not complete. A free parameter, θ_W, appears in addition to e because the symmetry group on which the model is based is a direct product of two simple groups, and would be unnecessary in a larger simple group.

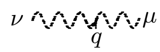
The Feynman rules for this model are obtained from (112)–(119) in the same way as in Chap. 8 for QCD. They are given in Fig. 9.1a–c.

Photon propagator



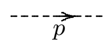
$$\frac{-\mathrm{i}g_{\mu\nu}}{q^2+\mathrm{i}\varepsilon}$$

$\mathrm{W}^{\pm}$, Z propagators



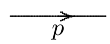
$$\frac{\mathrm{i}}{q^2-M^2+\mathrm{i}\varepsilon}[-g_{\mu\nu}+\frac{q_\mu q_\nu}{M^2}]$$

Higgs propagator



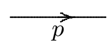
$$\frac{\mathrm{i}}{p^2-M_{\mathrm{H}}^2+\mathrm{i}\varepsilon}$$

Neutrino propagator



$$\frac{1-\gamma_5}{2}\frac{\mathrm{i}}{\not p+\mathrm{i}\varepsilon}\frac{1+\gamma_5}{2}$$

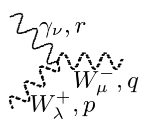
Lepton propagator



$$\frac{\mathrm{i}}{\not p-m_\ell+\mathrm{i}\varepsilon}$$

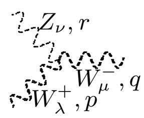
Fig. 9.1. (**a**) Propagators in the gauge-invariant model of one-lepton family

WWγ vertex



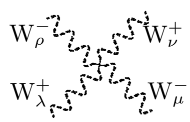
$$-\mathrm{i}e[(r-q)_\lambda g_{\mu\nu}+(q-p)_\nu g_{\lambda\mu} \\ +(p-r)_\mu g_{\lambda\nu}] \\ (p+r+q=0)$$

WWZ vertex



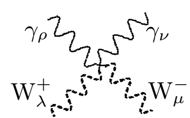
$$-\mathrm{i}g\cos\theta_{\mathrm{W}}[(r-q)_\lambda g_{\mu\nu}+(q-p)_\nu g_{\lambda\mu} \\ +(p-r)_\mu g_{\lambda\nu}] \\ (p+r+q=0)$$

W^4 vertex



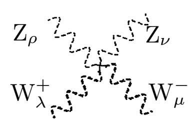
$$-\mathrm{i}g^2\,(g_{\lambda\rho}\,g_{\mu\nu}+g_{\lambda\mu}\,g_{\rho\nu}-2g_{\lambda\nu}\,g_{\mu\rho})$$

$\mathrm{W}^2\gamma\gamma$ vertex



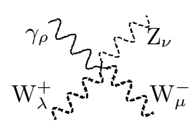
$$\mathrm{i}e^2(g_{\lambda\rho}\,g_{\mu\nu}+g_{\lambda\nu}\,g_{\rho\mu}-2g_{\lambda\mu}\,g_{\rho\nu})$$

$\mathrm{W}^2\mathrm{Z}^2$ vertex



$$\mathrm{i}(g\cos\theta_{\mathrm{W}})^2\,(g_{\lambda\rho}\,g_{\mu\nu}+g_{\lambda\nu}\,g_{\rho\mu} \\ -2g_{\lambda\mu}\,g_{\rho\nu})$$

$\mathrm{W}^2\mathrm{Z}\gamma$ vertex



$$\mathrm{i}eg\cos\theta_{\mathrm{W}}\,(g_{\lambda\rho}\,g_{\mu\nu}+g_{\lambda\nu}\,g_{\rho\mu} \\ -2g_{\lambda\mu}\,g_{\rho\nu})$$

Fig. 9.1. (**b**) Interaction vertices in the gauge-invariant model of one-lepton family: gauge boson self-couplings

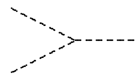
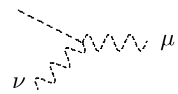
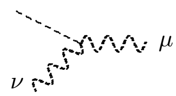
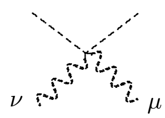
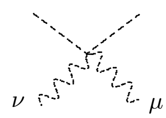
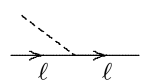
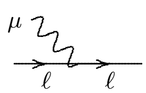
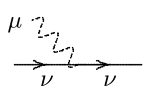
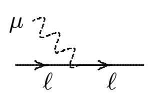
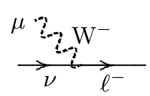
Three-Higgs vertex		$-\mathrm{i}\frac{3}{2}\, g\, \frac{M_\mathrm{H}^2}{M_\mathrm{W}}$
Four-Higgs vertex		$-\mathrm{i}\frac{3}{4}\, g^2\, \frac{M_\mathrm{H}^2}{M_\mathrm{W}^2}$
ZZ–Higgs vertex		$\mathrm{i}\, g\, \frac{M_\mathrm{W}}{\cos^2\theta_\mathrm{W}}\, g_{\mu\nu}$
WW–Higgs vertex		$\mathrm{i}gM_\mathrm{W}\, g_{\mu\nu}$
$\mathrm{W}^2\mathrm{H}^2$ vertex		$\mathrm{i}\, \frac{g^2}{2}\, g_{\mu\nu}$
$\mathrm{Z}^2\mathrm{H}^2$ vertex		$\mathrm{i}\, \frac{g^2}{2\cos^2\theta_\mathrm{W}}\, g_{\mu\nu}$
Higgs–lepton vertex		$-\mathrm{i}\, \frac{gm_\ell}{2M_\mathrm{W}}$
γee vertex		$\mathrm{i}\, e\, \gamma_\mu$
Z$\nu\nu$ vertex		$\frac{-\mathrm{i}\, g}{2\cos\theta_\mathrm{W}}\, \gamma_\mu\, \frac{1-\gamma_5}{2}$
Z$\ell\ell$ vertex		$\frac{-\mathrm{i}\, g}{2\cos\theta_\mathrm{W}}\, \gamma_\mu(g_\mathrm{V}^\ell - g_\mathrm{A}^\ell\gamma_5)$
W$\ell\nu$ vertex		$\frac{-\mathrm{i}\, g}{\sqrt{2}}\, \gamma_\mu\, \frac{1-\gamma_5}{2}$

Fig. 9.1. (**c**) Interaction vertices in the gauge-invariant model of one-lepton family: couplings in the Higgs boson and lepton sectors; $g_{A,V}^\ell$ are defined in Table 9.3.

9.3 Including u and d Quarks

We now introduce the u and d quarks, which form with the electron and the neutrino ν_e the first generation of fundamental fermions. The quarks and the leptons enter the model in rather similar ways, in spite of their distinctive characteristics. First, even though quarks are colored and leptons are not, no complications result because the electroweak interactions are insensitive to color. For this reason we will suppress the color label, with the understanding that the implied color indices are summed over where necessary. Second, quarks differ from leptons in their electric charges. However, the fact that $Q_u - Q_d = Q_\nu - Q_e = 1$ and the well-established observation that the weak charged currents of hadrons are left-handed suggest that the left chiral components of the quarks should be grouped, similarly to the leptons, into weak-isospin doublets. Finally, both u and d quarks are massive, whereas the neutrino is (believed to be) massless. This implies that the right chiral components of both quarks should appear in the model, to be compared with the sole e_R in the lepton sector. Therefore the quark sector should include a doublet ψ_L plus two singlets u_R and d_R in the weak-isospin group SU(2):

$$\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} ; \quad u_R , \quad d_R ; \tag{9.121}$$

and the model described by the Lagrangians in Sect. 9.2.2 should be amended to include the appropriate gauge-invariant terms for the quarks and the appropriate scalar–quark Yukawa couplings.

The Lagrangian for the free quarks is given by

$$\begin{aligned} \mathcal{L}_q^0 &= \overline{u}\, i\gamma^\mu \partial_\mu u + \overline{d}\, i\gamma^\mu \partial_\mu d \\ &= \overline{\psi}_L\, i\gamma^\mu \partial_\mu \psi_L + \overline{u}_R\, i\gamma^\mu \partial_\mu u_R + \overline{d}_R\, i\gamma^\mu \partial_\mu d_R . \end{aligned} \tag{9.122}$$

It is clearly invariant under global $SU_L(2) \times U_Y(1)$. To this symmetry correspond the conserved currents

$$j^i_\mu = \overline{\psi}_L \gamma_\mu \frac{\tau_i}{2} \psi_L , \qquad (i = 1, 2, 3) , \tag{9.123}$$

$$j^Y_\mu = Y_L\, \overline{\psi}_L \gamma_\mu \psi_L + Y_R^u\, \overline{u}_R \gamma_\mu u_R + Y_R^d\, \overline{d}_R \gamma_\mu d_R ; \tag{9.124}$$

and the conserved charges T_3 and Y are related as usual to the electric charge number Q through $Q = T_3 + \frac{1}{2} Y$. The values assigned to these quantum numbers for the u–d quark multiplets are listed in Table 9.2.

The $SU(2) \times U_Y(1)$ local gauge-invariant form of (122) is

$$\begin{aligned} \mathcal{L}_q &= \overline{\psi}_L i\gamma^\mu D^L_\mu \psi_L + \overline{u}_R i\gamma^\mu D^R_\mu u_R + \overline{d}_R i\gamma^\mu D^R_\mu d_R \\ &= \overline{\psi}_L i\gamma^\mu (\partial_\mu + ig\boldsymbol{A}_\mu + \tfrac{i}{2} g' Y_L B_\mu) \psi_L \\ &\quad + \overline{u}_R i\gamma^\mu (\partial_\mu + \tfrac{i}{2} g' Y_R^u B_\mu) u_R + \overline{d}_R i\gamma^\mu (\partial_\mu + \tfrac{i}{2} g' Y_R^d B_\mu) d_R . \end{aligned} \tag{9.125}$$

Table 9.2. Classification of the u–d quark family and assigned quantum numbers

	T	T_3	Y	Q
$\begin{pmatrix} u_{\mathrm{L}} \\ d_{\mathrm{L}} \end{pmatrix}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$ $-\frac{1}{3}$
u_{R}	0	0	$\frac{4}{3}$	$\frac{2}{3}$
d_{R}	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$

The scalar–quark interactions will include the couplings $(\overline{\psi}_{\mathrm{L}}\boldsymbol{\phi})d_{\mathrm{R}}$ and $\overline{d}_{\mathrm{R}}(\boldsymbol{\phi}^{\dagger}\psi_{\mathrm{L}})$ similar to those found in the lepton sector. In order to couple u_{R} to scalars in a gauge-invariant way, we also need φ^{-} and $\overline{\varphi}^{0}$, the charge conjugates to φ^{+} and φ^{0}, which form a doublet conjugate to $\boldsymbol{\phi}$, that is,

$$\boldsymbol{\phi}^{\mathrm{c}} = \mathrm{i}\tau_2\boldsymbol{\phi}^{*} = \begin{pmatrix} \overline{\varphi}^{0} \\ -\varphi^{-} \end{pmatrix} . \tag{9.126}$$

It has weak hypercharge $Y_{\mathrm{H^c}} = -Y_{\mathrm{H}} = -1$. The Yukawa quark couplings require two coupling constants, C_{u} and C_{d}, and assume the general form

$$\mathcal{L}_{\mathrm{qY}} = -C_{\mathrm{u}}\left[(\overline{\psi}_{\mathrm{L}}\boldsymbol{\phi}^{\mathrm{c}})u_{\mathrm{R}} + \overline{u}_{\mathrm{R}}(\boldsymbol{\phi}^{\mathrm{c}\dagger}\psi_{\mathrm{L}})\right] - C_{\mathrm{d}}\left[(\overline{\psi}_{\mathrm{L}}\boldsymbol{\phi})d_{\mathrm{R}} + \overline{d}_{\mathrm{R}}(\boldsymbol{\phi}^{\dagger}\psi_{\mathrm{L}})\right] . \tag{9.127}$$

Gauge invariance of these couplings under $U_{\mathrm{Y}}(1)$ is guaranteed by the assigned weak hypercharges of the particles: $Y_{\mathrm{L}} - Y_{\mathrm{R}}^{\mathrm{u}} = Y_{\mathrm{H^c}}$ and $Y_{\mathrm{L}} - Y_{\mathrm{R}}^{\mathrm{d}} = Y_{\mathrm{H}}$.

After breaking symmetry, one goes to the unitary gauge just as before, so that the scalar doublets become

$$\boldsymbol{\phi} \;\to \mathcal{S}\boldsymbol{\phi} \; = \tfrac{1}{\sqrt{2}}(v+H)\chi\,, \qquad \chi \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \tag{9.128}$$

$$\boldsymbol{\phi}^{\mathrm{c}} \to \mathcal{S}\boldsymbol{\phi}^{\mathrm{c}} = \tfrac{1}{\sqrt{2}}(v+H)\chi^{\mathrm{c}}\,, \qquad \chi^{\mathrm{c}} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9.129}$$

In the unitary gauge, the Yukawa interaction takes the form

$$\mathcal{L}_{\mathrm{qY}} = -\frac{1}{\sqrt{2}}(v+H)\left(C_{\mathrm{u}}\,\overline{u}u + C_{\mathrm{d}}\,\overline{d}d\right) , \tag{9.130}$$

which shows that through the Higgs mechanism the u and d quarks acquire the masses

$$m_{\mathrm{u}} = \tfrac{1}{\sqrt{2}}\,C_{\mathrm{u}}v \quad \text{and} \quad m_{\mathrm{d}} = \tfrac{1}{\sqrt{2}}\,C_{\mathrm{d}}v\,. \tag{9.131}$$

Inversely, the Yukawa couplings can be expressed in terms of the quark masses

$$C_{\mathrm{u}} = \frac{\sqrt{2}\,m_{\mathrm{u}}}{v} = \frac{gm_{\mathrm{u}}}{\sqrt{2}\,M_{\mathrm{W}}} \qquad \text{and} \qquad C_{\mathrm{d}} = \frac{\sqrt{2}\,m_{\mathrm{d}}}{v} = \frac{gm_{\mathrm{d}}}{\sqrt{2}\,M_{\mathrm{W}}}\,,$$

so that the Lagrangian $\mathcal{L}_{\mathrm{qY}}$ assumes the form

$$\mathcal{L}_{\mathrm{qY}} = -m_{\mathrm{u}}\,\overline{u}u - m_{\mathrm{d}}\,\overline{d}d - \frac{gm_{\mathrm{u}}}{2\,M_{\mathrm{W}}}\,\overline{u}u\,H - \frac{gm_{\mathrm{d}}}{2\,M_{\mathrm{W}}}\,\overline{d}d\,H\,. \tag{9.132}$$

The quark Lagrangian $\mathcal{L}_{\mathrm{q}}$ is very similar to the $\mathcal{L}_{\ell}$ considered in the last section. Written now in the unitary gauge, it includes, besides the usual kinetic terms for u and d fields, the following interaction terms. First, there are the contributions of the quark fields to the charged current interaction:

$$\begin{aligned}\mathcal{L}^{\mathrm{q}}_{\mathrm{cc}} &= -\tfrac{1}{\sqrt{2}}\,g\,\overline{\psi}_{\mathrm{L}}\gamma^{\mu}\left(\tau_{+}W_{\mu} + \tau_{-}W^{\dagger}_{\mu}\right)\psi_{\mathrm{L}} \\ &= -\tfrac{1}{\sqrt{2}}\,g\,\left(J^{\mu\dagger}W_{\mu} + J^{\mu}W^{\dagger}_{\mu}\right)\,;\end{aligned} \tag{9.133}$$

and then their contributions to the neutral current interaction:

$$\begin{aligned}\mathcal{L}^{\mathrm{q}}_{\mathrm{nc}} &= -\tfrac{1}{2}\,\overline{\psi}_{\mathrm{L}}\gamma_{\mu}\left(gA^{\mu}_{3}\tau_{3} + g'B^{\mu}Y_{\mathrm{L}}\right)\psi_{\mathrm{L}} - \tfrac{1}{2}\,g'Y^{i}_{\mathrm{R}}\,\overline{\psi}^{i}_{\mathrm{R}}\gamma_{\mu}\psi^{i}_{\mathrm{R}}\,B^{\mu} \\ &= -g\,j^{3}_{\mu}\,A^{\mu}_{3} - \tfrac{1}{2}\,g'\,j^{\mathrm{Y}}_{\mu}\,B^{\mu}\,,\end{aligned} \tag{9.134}$$

which are given in terms of the photon and the Z^{0} fields by

$$\begin{aligned}\mathcal{L}^{\mathrm{q}}_{\mathrm{nc}} &= -(gs_{\mathrm{W}}\,j^{3}_{\mu} + \tfrac{1}{2}\,g'c_{\mathrm{W}}\,j^{\mathrm{Y}}_{\mu})\,A^{\mu} - (gc_{\mathrm{W}}\,j^{3}_{\mu} - \tfrac{1}{2}\,g's_{\mathrm{W}}\,j^{\mathrm{Y}}_{\mu})\,Z^{\mu} \\ &= -e\,j^{\mathrm{em}}_{\mu}A^{\mu} - \frac{g}{c_{\mathrm{W}}}\,j^{\mathrm{Z}}_{\mu}Z^{\mu}\,.\end{aligned} \tag{9.135}$$

Here the Z-current for the u–d quarks may be written as

$$\begin{aligned}j^{\mathrm{Z}}_{\mu} &= j^{3}_{\mu} - s^{2}_{\mathrm{W}}\,j^{\mathrm{Y}}_{\mu} = \overline{\psi}\gamma_{\mu}(T_{3} - s^{2}_{\mathrm{W}}\,Q)\,\psi \\ &= \overline{\psi}_{\mathrm{L}}\gamma_{\mu}Z_{\mathrm{L}}\psi_{\mathrm{L}} + Z^{\mathrm{u}}_{\mathrm{R}}\,\overline{u}_{\mathrm{R}}\gamma_{\mu}u_{\mathrm{R}} + Z^{\mathrm{d}}_{\mathrm{R}}\,\overline{d}_{\mathrm{R}}\gamma_{\mu}d_{\mathrm{R}}\,.\end{aligned}$$

It preserves quark flavors and couples to both chiral components of quarks. Alternatively, it may be written as

$$j^{\mathrm{Z}}_{\mu} = \tfrac{1}{2}\,\overline{u}\gamma_{\mu}\left(g^{\mathrm{u}}_{\mathrm{V}} - g^{\mathrm{u}}_{\mathrm{A}}\gamma_{5}\right)u + \tfrac{1}{2}\,\overline{d}\gamma_{\mu}\left(g^{\mathrm{d}}_{\mathrm{V}} - g^{\mathrm{d}}_{\mathrm{A}}\gamma_{5}\right)d\,. \tag{9.136}$$

The weak charges and the weak neutral current coupling constants are defined as before, $Z_{\mathrm{L}} = T_{3} - s^{2}_{\mathrm{W}}\,Q$ and $Z_{\mathrm{R}} = -s^{2}_{\mathrm{W}}\,Q$, or alternatively, $g_{\mathrm{V}} = Z_{\mathrm{L}} + Z_{\mathrm{R}}$ and $g_{\mathrm{A}} = Z_{\mathrm{L}} - Z_{\mathrm{R}}$. They depend on a single parameter, the Weinberg angle θ_{W}. Their values are listed in Table 9.3.

Table 9.3. Charges and coupling constants of the weak neutral current for leptons and quarks in the standard model

f	Q	$Z^{\mathrm{f}}_{\mathrm{L}}$	$Z^{\mathrm{f}}_{\mathrm{R}}$	$g^{\mathrm{f}}_{\mathrm{V}}$	$g^{\mathrm{f}}_{\mathrm{A}}$
ν	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
e	-1	$-\frac{1}{2} + \sin^{2}\theta_{\mathrm{W}}$	$\sin^{2}\theta_{\mathrm{W}}$	$-\frac{1}{2} + 2\,\sin^{2}\theta_{\mathrm{W}}$	$-\frac{1}{2}$
u	$\frac{2}{3}$	$\frac{1}{2} - \frac{2}{3}\,\sin^{2}\theta_{\mathrm{W}}$	$-\frac{2}{3}\,\sin^{2}\theta_{\mathrm{W}}$	$\frac{1}{2} - \frac{4}{3}\sin^{2}\theta_{\mathrm{W}}$	$\frac{1}{2}$
d	$-\frac{1}{3}$	$-\frac{1}{2} + \frac{1}{3}\,\sin^{2}\theta_{\mathrm{W}}$	$\frac{1}{3}\,\sin^{2}\theta_{\mathrm{W}}$	$-\frac{1}{2} + \frac{2}{3}\sin^{2}\theta_{\mathrm{W}}$	$-\frac{1}{2}$

To close, we summarize the results found in this section. The unified model of the electroweak interaction for the first generation of fermions is described by the Lagrangian that includes (112)–(119) from the e–ν_e family and the following contributions from the u and d quarks:

$$\mathcal{L}_\mathrm{q} + \mathcal{L}_\mathrm{qY} = \mathcal{L}_\mathrm{q}^0 + \mathcal{L}_\mathrm{qH} + \mathcal{L}_\mathrm{cc}^\mathrm{q} + \mathcal{L}_\mathrm{nc}^\mathrm{q} \,. \tag{9.137}$$

The first term on the right-hand side gives the kinetic part

$$\mathcal{L}_\mathrm{q}^0 = \overline{u}\,(\mathrm{i}\gamma^\mu\partial_\mu - m_\mathrm{u})\,u + \overline{d}\,(\mathrm{i}\gamma^\mu\partial_\mu - m_\mathrm{d})d\,; \tag{9.138}$$

the second represents the couplings of quarks to the Higgs boson

$$\mathcal{L}_\mathrm{qH} = -\frac{gm_\mathrm{u}}{2\,M_\mathrm{W}}\,\overline{u}u\,H - \frac{gm_\mathrm{d}}{2\,M_\mathrm{W}}\,\overline{d}d\,H\,; \tag{9.139}$$

while the remaining terms represent the couplings of the gauge bosons to the charged and neutral currents for quarks:

$$\begin{aligned}\mathcal{L}_\mathrm{cc}^\mathrm{q} &= -\frac{g}{\sqrt{2}}\,\left(J_\mu^\dagger W^\mu + J_\mu W^{\mu\dagger}\right)\\ &= -\frac{g}{\sqrt{2}}\,\left(\overline{u}_\mathrm{L}\gamma_\mu d_\mathrm{L}\,W^\mu + \overline{d}_\mathrm{L}\gamma_\mu u_\mathrm{L}\,W^{\mu\dagger}\right)\,;\end{aligned} \tag{9.140}$$

$$\begin{aligned}\mathcal{L}_\mathrm{nc}^\mathrm{q} &= -\,e\,j_\mu^\mathrm{em}A^\mu - \frac{g}{c_\mathrm{W}}\,j_\mu^\mathrm{Z}Z^\mu\\ &= -\,e\left(\frac{2}{3}\,\overline{u}\gamma_\mu u - \frac{1}{3}\,\overline{d}\gamma_\mu d\right)A^\mu\\ &\quad - \frac{g}{2\,c_\mathrm{W}}\,\left[\overline{u}\gamma_\mu(g_\mathrm{V}^\mathrm{u} - g_\mathrm{A}^\mathrm{u}\,\gamma_5)u + \overline{d}\gamma_\mu(g_\mathrm{V}^\mathrm{d} - g_\mathrm{A}^\mathrm{d}\,\gamma_5)d\right]Z^\mu\,.\end{aligned} \tag{9.141}$$

A comparison with similar results for leptons in (112)–(119) shows the close parallel between the u, d quarks and the e, ν_e leptons in their electroweak interactions. Their charged current interactions are identical, and they both break parity in the strongest possible way. Their electromagnetic and neutral current interactions differ only because of their different electric and weak charges, and their couplings to the Higgs boson differ only in strengths because of their different masses.

Including u and d quarks adds two more parameters to the model, the quark masses, m_u and m_d. Thus, for one generation of quarks and leptons, the model requires seven independent parameters: in the original gauge-invariant Lagrangian, they are the two gauge couplings g and g', the two scalar self-couplings λ and μ^2, and the three Yukawa couplings C_e, C_u, and C_d; after symmetry breaking, they are replaced by e, θ_W, M_W, M_H, m_e, m_u, and m_d. The content of Fig. 9.1 is now complemented by the additional Feynman rules derived from (138)–(141) and listed in Fig. 9.2.

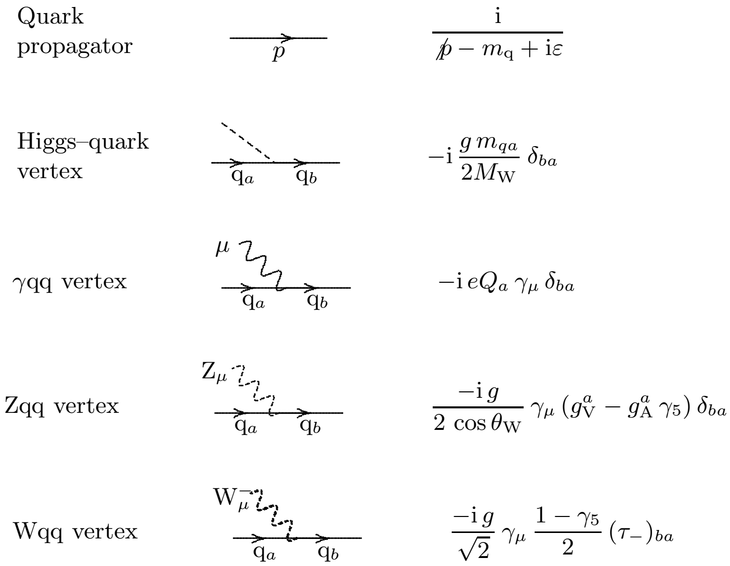


Fig. 9.2. Feynman diagrams for the electroweak interaction of quarks

9.4 Multigeneration Model

It is now a well-established experimental fact that there exist six leptons – e^-, ν_{e}, μ^-, ν_μ, τ^-, and ν_τ – and six quarks – u, d, c, s, t, and b – plus their corresponding antiparticles (cf. Table 7.9). Together they constitute the complete fermionic content of the standard model of the electroweak interactions. Although a casual look at the data on leptonic weak decays might indicate otherwise, the incorporation of all known leptons and quarks in the model involves much more than a mere replication of the formulation with a single family, as was presented in the last two sections for e^-, ν_{e}, u, and d. Over the years, observations of a multitude of weak processes have brought out many novel features; some have helped to shape the emerging theory, while others might yet find in it a possible explanation. These features include a certain mixing of the quark fields and the absence of a similar mixing of the lepton fields, the suppression of flavor-changing neutral currents and the phenomenon of CP violation.

9.4.1 The GIM Mechanism

The purely leptonic decay $\mu^- \to \mathrm{e}^-\overline{\nu}_{\mathrm{e}}\nu_\mu$ and the related scattering process $\mathrm{e}^-\nu_\mu \to \mu^-\nu_{\mathrm{e}}$ can be described to a good accuracy by the purely leptonic part, $L^\alpha L_\alpha^\dagger$, of the effective Hamiltonian (1),

$$\mathcal{H}_{\mathrm{weak}}^{\mathrm{l}} = \frac{G_{\mathrm{F}}}{\sqrt{2}}\,[\overline{\nu}_\mu\gamma^\alpha(1-\gamma_5)\mu]\,[\overline{e}\gamma_\alpha(1-\gamma_5)\nu_{\mathrm{e}}]\,. \tag{9.142}$$

By comparing the calculated μ^- lifetime, including radiative corrections, with the measured lifetime, $\tau_\mu = 2.179 \times 10^{-5}\,$s, one gets the value of the decay strength $G_{\rm F}$, as is given in (3).

On the other hand, the amplitudes of β-decays may be calculated with the semileptonic coupling terms, $L^\alpha H_\alpha^\dagger + L_\alpha^\dagger H^\alpha$, or

$$\mathcal{H}_{\rm weak}^{\rm sl} = \frac{G^{(0)}}{\sqrt{2}}\,[\overline{p}\gamma^\alpha(1 - c_{\rm A}\gamma_5)n]\,[\overline{e}\gamma_\alpha(1-\gamma_5)\nu_{\rm e}] + \text{h.c.}\,, \tag{9.143}$$

where, in order to fit data,

$$G^{(0)}/G_{\rm F} \approx 0.975\,, \qquad c_{\rm A} = 1.2573 \pm 0.0028\,, \tag{9.144}$$

both of which are close to but unmistakably different from 1. The apparent similarity between (142) and (143) suggests there should be universality in the structure of interactions at the quark level,

$$\mathcal{H}_{\rm weak} = \frac{G^{(0)}}{\sqrt{2}}\,[\overline{u}\gamma^\alpha(1-\gamma_5)d]\,[\overline{e}\gamma_\alpha(1-\gamma_5)\nu_{\rm e}] + \text{h.c.}\,. \tag{9.145}$$

While it is expected that the *axial-vector* current coupling gets modified by the QCD effects when hadrons are involved, causing $c_{\rm A}$ to deviate from 1, there is a strong belief that, just as the electromagnetic current is conserved, so too is the weak *vector* current, so that the matrix element of this current should not get renormalized at the hadronic level: conservation of vector current, $\partial^\mu(\overline{u}\gamma_\mu d) = 0$, implies no renormalization by strong interactions, $\langle p\,|\,\overline{u}\gamma_\mu d\,|\,n\rangle = \overline{p}\gamma_\mu n$, at zero invariant momentum transfer. Therefore, the deviation of $G^{(0)}$ from $G_{\rm F}$ must be a genuine effect, persisting even after radiative corrections are taken into account, and thus must have a deeper physical origin: these strangeness-conserving charged quark currents alone cannot generate an SU(2) group in the way the lepton currents do.

The general structure of the weak interaction is found to be of the V–A type, as in (142) and (143), but the strength of the strangeness-changing decay, $G^{(1)}$, is consistently smaller than that of the β-decay: $G^{(1)} \approx 0.22\,G_{\rm F}$ (from comparing, for example, the rate of $\Lambda \to {\rm pe}^-\bar{\nu}$ with that of ${\rm n} \to {\rm pe}^-\bar{\nu}$). To reflect this fact, one introduces a parameter $\theta_{\rm C}$ (called the Cabibbo mixing angle), such that $\sin\theta_{\rm C} = G^{(1)}/G_{\rm F} \approx 0.22$ may give a measure of the relative strength of the strangeness-changing charged current. The amplitudes of strangeness-changing decays can be obtained from the *charged current*

$$H_\mu^\dagger = \overline{u}\gamma_\mu(1-\gamma_5)d_{\rm C}\,, \tag{9.146}$$

which differs from the corresponding current in (145) by replacing d with

$$d_{\rm C} = d\,\cos\theta_{\rm C} + s\,\sin\theta_{\rm C}\,, \tag{9.147}$$

where s is the strange quark field. However, a similar substitution $d \to d_{\mathrm{C}}$ in the *neutral current* j_μ^{Z} of (136) would give

$$\bar{d}_{\mathrm{C}}\, \gamma_\mu\, (g_{\mathrm{V}}^{\mathrm{d}} - g_{\mathrm{A}}^{\mathrm{d}} \gamma_5)\, d_{\mathrm{C}}\,, \tag{9.148}$$

and would lead to terms like

$$\cos\theta_{\mathrm{C}}\, \sin\theta_{\mathrm{C}}\, \left[\, \bar{d}\, \gamma_\mu\, (g_{\mathrm{V}}^{\mathrm{d}} - g_{\mathrm{A}}^{\mathrm{d}} \gamma_5)\, s + \overline{s}\, \gamma_\mu\, (g_{\mathrm{V}}^{\mathrm{d}} - g_{\mathrm{A}}^{\mathrm{d}} \gamma_5)\, d \,\right] . \tag{9.149}$$

The neutral flavor-changing transition $\mathrm{s} \to \mathrm{d}$ (as in $\mathrm{K}^- \to \pi^- \mathrm{e}^+ \mathrm{e}^-$) would then be possible at a strength roughly comparable to that of $\mathrm{s} \to \mathrm{u}$ processes (as in $\mathrm{K}^- \to \pi^0 \mathrm{e}^- \bar{\nu}$), in sharp disagreement with data. Again, the neutral current in this form does not seem to be complete.

Glashow, Iliopoulos, and Maiani (GIM) suggested in 1970 that an additional flavor, the *charm* c, should exist and should form with s_{C}, the orthogonal complement to d_{C}, a second quark doublet. The lower components of the two doublets $(u,\, d_{\mathrm{C}})$ and $(c,\, s_{\mathrm{C}})$ are related to the physical quarks d and s by an orthogonal transformation,

$$\begin{pmatrix} d_{\mathrm{C}} \\ s_{\mathrm{C}} \end{pmatrix} = \boldsymbol{V}_{\mathrm{C}} \begin{pmatrix} d \\ s \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_{\mathrm{C}} & \sin\theta_{\mathrm{C}} \\ -\sin\theta_{\mathrm{C}} & \cos\theta_{\mathrm{C}} \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} . \tag{9.150}$$

Then, as long as the coupling strengths of the two doublets to Z_μ are equal, the flavor-changing neutral processes, as in $\mathrm{s} \to \mathrm{d}$, should be suppressed to all orders of perturbation because of the orthogonality of the Cabibbo rotation, $\boldsymbol{V}_{\mathrm{C}}^{\mathrm{T}} \boldsymbol{V}_{\mathrm{C}} = 1$, so that

$$\bar{d}_{\mathrm{C}} d_{\mathrm{C}} + \overline{s}_{\mathrm{C}} s_{\mathrm{C}} = \bar{d} d + \overline{s} s\,. \tag{9.151}$$

To sum up, the contributions of quarks to the charged and neutral currents that are needed to reproduce the $\Delta S = 0$ and $|\Delta S| = 1$ weak transitions at low energies should have the forms

$$J_\mu(\text{quarks}) = \tfrac{1}{2}\, \bar{d}_{\mathrm{C}}\, \gamma_\mu\, (1 - \gamma_5) u + \tfrac{1}{2}\, \overline{s}_{\mathrm{C}}\, \gamma_\mu (1 - \gamma_5) c\,, \tag{9.152}$$

and

$$\begin{aligned} j_\mu^{\mathrm{Z}}(\text{quarks}) = {} & \tfrac{1}{2}\, \bar{u}\, \gamma_\mu\, (g_{\mathrm{V}}^{\mathrm{u}} - g_{\mathrm{A}}^{\mathrm{u}} \gamma_5)\, u + \tfrac{1}{2}\, \bar{d}\, \gamma_\mu (g_{\mathrm{V}}^{\mathrm{d}} - g_{\mathrm{A}}^{\mathrm{d}} \gamma_5)\, d \\ & + \tfrac{1}{2}\, \bar{c} \gamma_\mu\, (g_{\mathrm{V}}^{\mathrm{u}} - g_{\mathrm{A}}^{\mathrm{u}} \gamma_5)\, c + \tfrac{1}{2}\, \bar{s}\, \gamma_\mu (g_{\mathrm{V}}^{\mathrm{d}} - g_{\mathrm{A}}^{\mathrm{d}} \gamma_5)\, s\,. \end{aligned} \tag{9.153}$$

Note that u and c have been assigned equal weak charges; similarly, d and s. The forms of these currents indicate that the left chiral components of fields should be considered weak-isospin doublets, $(u_{\mathrm{L}},\, d_{\mathrm{CL}})$ and $(c_{\mathrm{L}},\, s_{\mathrm{CL}})$, while all right chiral components, u_{R}, d_{R}, c_{R}, and s_{R}, weak-isospin singlets. The presence of a mixing angle in the left-handed quark sectors means that a clear distinction must be made between *gauge symmetry eigenstates* (also referred to as *weak interaction eigenstates*), d_{C} and s_{C}, endowed with definite gauge transformation properties, and *mass eigenstates*, d and s, having definite masses acquired through spontaneous symmetry breaking. As u_{R} and d_{R} do not couple to c_{R} or s_{R}, no mixing occurs among the right-handed quarks.

9.4.2 Classification Scheme for Fermions

When all known leptons and quarks are introduced into the model, we must similarly distinguish between the gauge symmetry basis and the physical (mass) basis. It is the gauge symmetry eigenstates (which will be marked by a prime accent, as in f') that describe the fermionic content of the gauge-invariant model. Just as in the one-family model, so too in the general model the left chiral components of fields transform as isodoublets and the right chiral components transform as isosinglets. They are shown in Table 9.4 together with their quantum numbers in $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}_\mathrm{Y}(1)$. We denote by $\boldsymbol{l}$ and $\boldsymbol{q}$ the vectors in generation space, with components $\ell_{A\mathrm{L}}$ and $q_{A\mathrm{L}}$, for $A = 1, 2, 3$, designating the three lepton and quark doublets. We will also need vectors $\boldsymbol{\nu}'$, $\boldsymbol{e}'$, $\boldsymbol{u}'$, and $\boldsymbol{d}'$, which have as components lepton or quark fields of equal charges:

$$\begin{aligned}
\boldsymbol{\nu}' &= (\nu'_\mathrm{e},\, \nu'_\mu,\, \nu'_\tau\,)\,; \\
\boldsymbol{e}' &= (e',\, \mu',\, \tau'\,)\,; \\
\boldsymbol{u}' &= (u',\, c',\, t'\,)\,; \\
\boldsymbol{d}' &= (d',\, s',\, b'\,)\,.
\end{aligned} \tag{9.154}$$

Table 9.4. SU(2)×U(1) classification and quantum numbers of the fundamental fermions in the standard model

	1	2	3	T	T_3	Y	Q
$\ell_{A\mathrm{L}}$	$\begin{pmatrix}\nu'_{\mathrm{eL}}\\ e'_\mathrm{L}\end{pmatrix}$	$\begin{pmatrix}\nu'_{\mu\mathrm{L}}\\ \mu'_\mathrm{L}\end{pmatrix}$	$\begin{pmatrix}\nu'_{\tau\mathrm{L}}\\ \tau'_\mathrm{L}\end{pmatrix}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	-1	$\begin{matrix}0\\ -1\end{matrix}$
$e'_{A\mathrm{L}}$	e'_R	μ'_R	τ'_R	0	0	-2	-1
$q_{A\mathrm{L}}$	$\begin{pmatrix}u'_\mathrm{L}\\ d'_\mathrm{L}\end{pmatrix}$	$\begin{pmatrix}c'_\mathrm{L}\\ s'_\mathrm{L}\end{pmatrix}$	$\begin{pmatrix}t'_\mathrm{L}\\ b'_\mathrm{L}\end{pmatrix}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{1}{3}$	$\begin{matrix}\frac{2}{3}\\ -\frac{1}{3}\end{matrix}$
$u'_{A\mathrm{R}}$	u'_R	c'_R	t'_R	0	0	$\frac{4}{3}$	$\frac{2}{3}$
$d'_{A\mathrm{R}}$	d'_R	s'_R	b'_R	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$

9.4.3 Fermion Families and the CKM Matrix

The incorporation of additional fermions in the model leaves the gauge and scalar sectors unchanged. It only affects the fermion–gauge and fermion–scalar couplings. The gauge-invariant fermion Lagrangian now reads:

$$\begin{aligned}
\mathcal{L}_\mathrm{F} &= \overline{\boldsymbol{\psi}}_\mathrm{L} \mathrm{i}\gamma^\mu\, D^\mathrm{L}_\mu \boldsymbol{\psi}_\mathrm{L} + \overline{\boldsymbol{\psi}}_\mathrm{R} \mathrm{i}\gamma^\mu\, D^\mathrm{R}_\mu \boldsymbol{\psi}_\mathrm{R} \\
&= \overline{\ell}_{A\mathrm{L}} \mathrm{i}\gamma^\mu\, D^\mathrm{L}_\mu \ell_{A\mathrm{L}} + \overline{e}'_{A\mathrm{R}} \mathrm{i}\gamma^\mu\, D^\mathrm{R}_\mu e'_{A\mathrm{R}} \\
&\quad + \overline{q}_{A\mathrm{L}} \mathrm{i}\gamma^\mu\, D^\mathrm{L}_\mu q_{A\mathrm{L}} + \overline{u}'_{A\mathrm{R}} \mathrm{i}\gamma^\mu\, D^\mathrm{R}_\mu u'_{A\mathrm{R}} + \overline{d}'_{A\mathrm{R}} \mathrm{i}\gamma^\mu\, D^\mathrm{R}_\mu d'_{A\mathrm{R}}\,,
\end{aligned} \tag{9.155}$$

where the covariant derivatives of fields are

$$
\begin{aligned}
D_\mu^{\mathrm{L}}\ell_{A\mathrm{L}} &= (\partial_\mu + \mathrm{i}\tfrac{1}{2}g\tau_i A_\mu^i - \mathrm{i}\tfrac{1}{2}g' B_\mu)\,\ell_{A\mathrm{L}}\,,\\
D_\mu^{\mathrm{R}}e'_{A\mathrm{R}} &= (\partial_\mu - \mathrm{i}g' B_\mu)\,e'_{A\mathrm{R}}\,,\\
D_\mu^{\mathrm{L}}q_{A\mathrm{L}} &= (\partial_\mu + \mathrm{i}\tfrac{1}{2}g\tau_i A_\mu^i + \mathrm{i}\tfrac{1}{6}g' B_\mu)\,q_{A\mathrm{L}}\,,\\
D_\mu^{\mathrm{R}}u'_{A\mathrm{R}} &= (\partial_\mu + \mathrm{i}\tfrac{2}{3}g' B_\mu)\,u'_{A\mathrm{R}}\,,\\
D_\mu^{\mathrm{R}}d'_{A\mathrm{R}} &= (\partial_\mu - \mathrm{i}\tfrac{1}{3}g' B_\mu)\,d'_{A\mathrm{R}}\,. \qquad (9.156)
\end{aligned}
$$

With all the neutrinos assumed to be exactly massless, the Yukawa couplings for the remaining fermions are

$$
\begin{aligned}
\mathcal{L}_{\mathrm{Y}} = -\,\big[&C^{\mathrm{e}}_{AB}\,(\bar{\ell}_{A\mathrm{L}}\boldsymbol{\phi})\,e'_{B\mathrm{R}}\\
&+ C^{\mathrm{u}}_{AB}\,(\bar{q}_{A\mathrm{L}}\boldsymbol{\phi}^{\mathrm{c}})\,u'_{B\mathrm{R}} + C^{\mathrm{d}}_{AB}\,(\bar{q}_{A\mathrm{L}}\boldsymbol{\phi})\,d'_{B\mathrm{R}} + \text{h.c.}\big]\,. \qquad (9.157)
\end{aligned}
$$

The coupling strengths are given by three 3×3 matrices $\boldsymbol{C}^f$, with $f = \mathrm{e}$, u, d, one matrix for each set of equally charged fermions.

Fermion Mass Matrix. Upon spontaneous symmetry breaking and going to the usual unitary gauge, which implies in particular

$$
\begin{aligned}
\boldsymbol{\phi} &\to \tfrac{1}{\sqrt{2}}\,(v+H)\chi\,,\\
\boldsymbol{\phi}^{\mathrm{c}} &\to \tfrac{1}{\sqrt{2}}\,(v+H)\chi^{\mathrm{c}}\,, \qquad (9.158)
\end{aligned}
$$

the various Yukawa couplings become

$$
\begin{aligned}
(\bar{\ell}_{A\mathrm{L}}\boldsymbol{\phi}) &= \tfrac{1}{\sqrt{2}}\,(v+H)\,\bar{e}'_{A\mathrm{L}}\,,\\
(\bar{q}_{A\mathrm{L}}\boldsymbol{\phi}) &= \tfrac{1}{\sqrt{2}}\,(v+H)\,\bar{d}'_{A\mathrm{L}}\,,\\
(\bar{q}_{A\mathrm{L}}\boldsymbol{\phi}^{\mathrm{c}}) &= \tfrac{1}{\sqrt{2}}\,(v+H)\,\bar{u}'_{A\mathrm{L}}\,.
\end{aligned}
$$

The corresponding terms contribute to the Yukawa Lagrangian

$$
\mathcal{L}_{\mathrm{Y}} = -\left(1 + \frac{H}{v}\right)\left(\overline{\boldsymbol{e}}'_{\mathrm{L}}\mathcal{M}'_{\mathrm{e}}\boldsymbol{e}'_{\mathrm{R}} + \overline{\boldsymbol{u}}'_{\mathrm{L}}\mathcal{M}'_{\mathrm{u}}\boldsymbol{u}'_{\mathrm{R}} + \overline{\boldsymbol{d}}'_{\mathrm{L}}\mathcal{M}'_{\mathrm{d}}\boldsymbol{d}'_{\mathrm{R}} + \text{h.c.}\right), \quad (9.159)
$$

which is expressed in terms of the fermionic mass matrices in the gauge eigenstate basis

$$
\mathcal{M}'_f = \frac{v}{\sqrt{2}}\,\boldsymbol{C}^f\,, \quad \text{for} \quad f = \mathrm{e},\ \mathrm{u},\ \mathrm{d}\,. \qquad (9.160)
$$

These matrices are generally neither symmetric nor Hermitian, but they can still be diagonalized by biunitary transformations. For each fundamental massive fermion f, one can write $\mathcal{M}'_f$ as the product of a Hermitian matrix $\boldsymbol{H}$ and a unitary matrix $\boldsymbol{T}$ as follows:

$$
\mathcal{M}'_f = \boldsymbol{H}_f\,\boldsymbol{T}_f\,; \quad \boldsymbol{H}_f = \sqrt{\mathcal{M}'_f\mathcal{M}'^{\dagger}_f}\,.
$$

Given that by construction $\boldsymbol{H}_f$ is Hermitian and positive-definite, the matrix $\boldsymbol{T}_f$ can be shown to be unitary, and $\boldsymbol{H}_f$ can be diagonalized by another unitary matrix, $\boldsymbol{S}_f$,

$$\boldsymbol{S}_f \boldsymbol{H}_f \boldsymbol{S}_f^\dagger = \mathcal{M}_f \,, \tag{9.161}$$

so that

$$\mathcal{M}'_f = \boldsymbol{S}_f^\dagger \mathcal{M}_f \boldsymbol{S}_f \, \boldsymbol{T}_f \,. \tag{9.162}$$

This means that for each term in (159) one may transform the mass matrix written there in the gauge eigenstate basis, $\mathcal{M}'_f$, into a diagonal matrix in the mass eigenstate basis, $\mathcal{M}_f$,

$$\overline{\boldsymbol{\psi}}'_{f\mathrm{L}} \mathcal{M}'_f \boldsymbol{\psi}'_{f\mathrm{R}} = \overline{\boldsymbol{\psi}}_{f\mathrm{L}} \mathcal{M}_f \boldsymbol{\psi}_{f\mathrm{R}} \,, \quad \text{for} \quad f = \mathrm{e},\ \mathrm{u},\ \mathrm{d} \,. \tag{9.163}$$

Here the *mass eigenstates* $\boldsymbol{\psi}_i$ are related to the *gauge eigenstates* $\boldsymbol{\psi}'_i$ by linear transformations:

$$\begin{aligned} \boldsymbol{\psi}_{f\mathrm{L}} &\equiv \boldsymbol{B}_{f\mathrm{L}} \boldsymbol{\psi}'_{f\mathrm{L}} = \boldsymbol{S}_f \boldsymbol{\psi}'_{f\mathrm{L}} \,, \\ \boldsymbol{\psi}_{f\mathrm{R}} &\equiv \boldsymbol{B}_{f\mathrm{R}} \boldsymbol{\psi}'_{f\mathrm{R}} = \boldsymbol{S}_f \boldsymbol{T}_f \, \boldsymbol{\psi}'_{f\mathrm{R}} \,, \end{aligned} \tag{9.164}$$

and the matrix $\mathcal{M}_f$ is diagonal,

$$(\mathcal{M}_f)_{AB} = m_A^f \, \delta_{AB} \,, \tag{9.165}$$

with the diagonal elements identified with the masses of the nine massive fermions emerging from the model:

$$\begin{aligned} \mathcal{M}_\mathrm{e} &= \mathrm{diagonal}\,(m_1^\mathrm{e},\, m_2^\mathrm{e},\, m_3^\mathrm{e}) = \mathrm{diagonal}\,(m_\mathrm{e},\, m_\mu,\, m_\tau) \,, \\ \mathcal{M}_\mathrm{u} &= \mathrm{diagonal}\,(m_1^\mathrm{u},\, m_2^\mathrm{u},\, m_3^\mathrm{u}) = \mathrm{diagonal}\,(m_\mathrm{u},\, m_\mathrm{c},\, m_\mathrm{t}) \,, \\ \mathcal{M}_\mathrm{d} &= \mathrm{diagonal}\left(m_1^\mathrm{d},\, m_2^\mathrm{d},\, m_3^\mathrm{d}\right) = \mathrm{diagonal}\,(m_\mathrm{d},\, m_\mathrm{s},\, m_\mathrm{b}) \,. \end{aligned} \tag{9.166}$$

The Yukawa Lagrangian now assumes a simpler form

$$\mathcal{L}_\mathrm{Y} = -\left(1 + \frac{H}{v}\right) \left[m_A^\mathrm{e} \left(\bar{e}_A e_A\right) + m_A^\mathrm{u} \left(\bar{u}_A u_A\right) + m_A^\mathrm{d} \left(\bar{d}_A d_A\right) \right] \,, \tag{9.167}$$

where, as before, $v = 2M_\mathrm{W}/g$. The strengths of the coupling of fermions to the Higgs depend linearly on m_A^f/M_W, a factor which ranges from 6×10^{-6} for the electron to 2 for the top quark.

Fermion Currents. Let us turn now to the fermion Lagrangian (155). Exactly as in the situation with one generation, here too the interaction terms describe the couplings of the gauge fields to the neutral and charged currents, which now, however, involve all quarks and leptons. The *electromagnetic current* is

$$\begin{aligned} j_\mu^{\mathrm{em}} &= \overline{\boldsymbol{e}}'\gamma_\mu Q\boldsymbol{e}' + \overline{\boldsymbol{u}}'\gamma_\mu Q\boldsymbol{u}' + \overline{\boldsymbol{d}}'\gamma_\mu Q\boldsymbol{d}' \\ &= \overline{\boldsymbol{e}}\gamma_\mu \boldsymbol{B}_{\mathrm{e}} Q \boldsymbol{B}_{\mathrm{e}}^\dagger \boldsymbol{e} + \overline{\boldsymbol{u}}\gamma_\mu \boldsymbol{B}_{\mathrm{u}} Q \boldsymbol{B}_{\mathrm{u}}^\dagger \boldsymbol{u} + \overline{\boldsymbol{d}}\gamma_\mu \boldsymbol{B}_{\mathrm{d}} Q \boldsymbol{B}_{\mathrm{d}}^\dagger \boldsymbol{d}\,. \end{aligned} \tag{9.168}$$

Since fermions with the same charge and the same helicity have the same transformation properties under the gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)$, the matrices $\boldsymbol{B}_{f\mathrm{L}}$ and $\boldsymbol{B}_{f\mathrm{R}}$ commute with the charge operator Q. And since both matrices are unitary, one immediately has

$$\boldsymbol{B}_{f\mathrm{L}} Q \boldsymbol{B}_{f\mathrm{L}}^\dagger = Q\,, \qquad \boldsymbol{B}_{f\mathrm{R}} Q \boldsymbol{B}_{f\mathrm{R}}^\dagger = Q\,. \tag{9.169}$$

In other words, in each case, $Q = T_3 + \frac{1}{2}Y$ is proportional to the identity matrix, with the same proportionality coefficients in both bases. Therefore,

$$j_\mu^{\mathrm{em}} = \overline{\boldsymbol{e}}\gamma_\mu Q\boldsymbol{e} + \overline{\boldsymbol{u}}\gamma_\mu Q\boldsymbol{u} + \overline{\boldsymbol{d}}\gamma_\mu Q\boldsymbol{d}\,. \tag{9.170}$$

By the same token, the weak *neutral current*

$$j_\mu^{\mathrm{Z}} = \overline{\boldsymbol{\nu}}'_{\mathrm{L}}\gamma_\mu Z\boldsymbol{\nu}'_{\mathrm{L}} + \overline{\boldsymbol{e}}'\gamma_\mu Z\boldsymbol{e}' + \overline{\boldsymbol{u}}'\gamma_\mu Z\boldsymbol{u}' + \overline{\boldsymbol{d}}'\gamma_\mu Z\boldsymbol{d}' \tag{9.171}$$

is unchanged in form when written in the mass eigenstate basis:

$$j_\mu^{\mathrm{Z}} = \overline{\boldsymbol{\nu}}_{\mathrm{L}}\gamma_\mu Z\boldsymbol{\nu}_{\mathrm{L}} + \overline{\boldsymbol{e}}\gamma_\mu Z\boldsymbol{e} + \overline{\boldsymbol{u}}\gamma_\mu Z\boldsymbol{u} + \overline{\boldsymbol{d}}\gamma_\mu Z\boldsymbol{d}\,. \tag{9.172}$$

This is again because $\boldsymbol{B}_{f\mathrm{L}}$ and $\boldsymbol{B}_{f\mathrm{R}}$ are unitary and commute with the weak charge operator Z:

$$\boldsymbol{B}_{f\mathrm{L}}(T_3 - Qs_{\mathrm{W}}^2)\boldsymbol{B}_{f\mathrm{L}}^\dagger = T_3 - Qs_{\mathrm{W}}^2\,, \quad \boldsymbol{B}_{f\mathrm{R}}(Qs_{\mathrm{W}}^2)\boldsymbol{B}_{f\mathrm{R}}^\dagger = Qs_{\mathrm{W}}^2\,. \tag{9.173}$$

Thus, the crucial property that the neutral currents are flavor-diagonal survives intact the transformation of basis; each chiral component of fermion goes to itself after emitting or absorbing a Z_μ. Note that in the above we have defined $\boldsymbol{\nu}_{\mathrm{L}} \equiv \boldsymbol{S}_\nu \boldsymbol{\nu}'_{\mathrm{L}}$ for any arbitrary unitary matrix, arbitrary because it is not constrained by any mass matrix since the neutrinos are assumed to be mass-degenerate, i.e. massless.

Let us now consider the *charged current*

$$\begin{aligned} J_\mu^\dagger = j_\mu^1 + \mathrm{i}j_\mu^2 &= \overline{\boldsymbol{l}}_{\mathrm{L}}\gamma_\mu\tau_+\boldsymbol{l}_{\mathrm{L}} + \overline{\boldsymbol{q}}_{\mathrm{L}}\gamma_\mu\tau_+\boldsymbol{q}_{\mathrm{L}} \\ &= \overline{\boldsymbol{\nu}}'_{\mathrm{L}}\gamma_\mu\boldsymbol{e}'_{\mathrm{L}} + \overline{\boldsymbol{u}}'_{\mathrm{L}}\gamma_\mu\boldsymbol{d}'_{\mathrm{L}}\,. \end{aligned}$$

In the lepton sector,

$$\overline{\boldsymbol{\nu}'}_{\mathrm{L}}\gamma_\mu\boldsymbol{e}'_{\mathrm{L}} = \overline{\boldsymbol{\nu}}_{\mathrm{L}}\,\gamma_\mu\,\boldsymbol{S}_\nu\boldsymbol{S}_{\mathrm{e}}^\dagger\,\boldsymbol{e}_{\mathrm{L}}\,. \tag{9.174}$$

Since $\boldsymbol{S}_\nu$ is an arbitrary unitary matrix, as we have already noted, it may be chosen so that $\boldsymbol{V}_\ell \equiv \boldsymbol{S}_\nu \boldsymbol{S}_\mathrm{e}^\dagger = 1$. This convention is allowed as long as the neutrinos remain exactly massless; but if it turns out that some or all of them acquire a nonnegligible mass, then $\boldsymbol{V}_\ell \neq 1$ necessarily (see Chap. 12). With $\boldsymbol{V}_\ell = 1$ so chosen, the lepton term reduces to

$$\overline{\boldsymbol{\nu}'}_\mathrm{L} \gamma_\mu \boldsymbol{e}'_\mathrm{L} = \overline{\boldsymbol{\nu}}_\mathrm{L} \gamma_\mu \boldsymbol{e}_\mathrm{L} = \overline{\nu}_{A\mathrm{L}} \gamma_\mu e_{A\mathrm{L}} \,.$$

In other words, it remains the same in form in the mass eigenstate basis and is diagonal in the generation labels.

As for the quarks, they contribute to the charged current

$$\overline{\boldsymbol{u}}'_\mathrm{L} \gamma_\mu \boldsymbol{d}'_\mathrm{L} = \overline{\boldsymbol{u}}_\mathrm{L} \gamma_\mu \, \boldsymbol{S}_\mathrm{u} \boldsymbol{S}_\mathrm{d}^\dagger \, \boldsymbol{d}_\mathrm{L} \,. \tag{9.175}$$

Since $\boldsymbol{S}_\mathrm{u} \boldsymbol{S}_\mathrm{d}^\dagger \neq 1$ generally, different generations are all mixed up in the quark mass eigenstates. The mixing may be entirely limited to either the u-type quarks or the d-type quarks. But it is customary to leave the three quarks of charge $Q = 2/3$ unmixed and let all the mixing be confined to the $Q = -1/3$ charge sector. Accordingly, with the shorthand notation

$$\boldsymbol{d}'' \equiv \boldsymbol{S}_\mathrm{u} \boldsymbol{S}_\mathrm{d}^\dagger \, \boldsymbol{d} \,,$$

the complete charged current for the model becomes

$$J_\mu^\dagger = \bar{\boldsymbol{\nu}}_\mathrm{L} \gamma_\mu \boldsymbol{e}_\mathrm{L} + \bar{\boldsymbol{u}}_\mathrm{L} \gamma_\mu \boldsymbol{d}''_\mathrm{L} \,. \tag{9.176}$$

Through this current, a neutrino converts itself into its corresponding charged lepton, conserving the lepton type, whereas a u-type quark can couple to any flavor of the d-type quarks, which results in a far greater variety of hadronic weak processes.

The unitary matrix $\boldsymbol{V} \equiv \boldsymbol{S}_\mathrm{u} \boldsymbol{S}_\mathrm{d}^\dagger$ is the generalization to three quark families of the Cabibbo rotation matrix. It was first introduced by Kobayashi and Maskawa (1973) and for this reason is referred to as the Cabibbo–Kobayashi–Maskawa (CKM) matrix. Explicitly,

$$\begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = \begin{pmatrix} V_\mathrm{ud} & V_\mathrm{us} & V_\mathrm{ub} \\ V_\mathrm{cd} & V_\mathrm{cs} & V_\mathrm{cb} \\ V_\mathrm{td} & V_\mathrm{ts} & V_\mathrm{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \,. \tag{9.177}$$

In general, a unitary $N \times N$ matrix can be parameterized by N^2 independent real quantities ($2N^2$ real parameters minus N^2 unitarity relations). Of these, $N(N-1)/2$ may be taken as the Euler angles associated with rotations in N-dimensional space. The remaining $N(N+1)/2$ are called phases, not all of which have physical meaning as some may be removed by redefining the quark fields that form the basis of the matrix representation. Of these $2N$ field phases (N from the up-type quarks and another N from the down-type quarks), $2N-1$ are not measurable. Thus, the number of measurable phases in the matrix is $\frac{1}{2}\,N(N+1)-(2N-1)=\frac{1}{2}\,(N-1)(N-2)$. In the standard model, with $N=3$ quark families, $\boldsymbol{V}$ contains 3 angles and 1 phase. A complex mixing matrix of this kind provides a mechanism for CP violation. Since no such a phase can appear in the presence of only two families of quarks, the existence of a third family could have been inferred from the observed CP violation in the neutral K mesons before the actual discoveries of the b and t quarks. Various equivalent parameterizations of $\boldsymbol{V}$ are possible; a popular one is

$$\boldsymbol{V}=\begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}\,\mathrm{e}^{-\mathrm{i}\delta_{13}} \\ -s_{12}\,c_{23}-c_{12}\,s_{23}\,s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{12}\,c_{23}-s_{12}\,s_{23}\,s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & s_{23}\,c_{13} \\ s_{12}\,s_{23}-c_{12}\,c_{23}\,s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & -c_{12}\,s_{23}-s_{12}\,c_{23}\,s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{23}\,c_{13} \end{pmatrix}. \quad (9.178)$$

Here $c_{AB}=\cos\theta_{AB}$ and $s_{AB}=\sin\theta_{AB}$, with $A,B=1,2,3$ being generation labels. The real angles θ_{12}, θ_{13}, and θ_{23} can be made to lie in the first quadrant by properly choosing the quark field phases; then $c_{AB}\geq 0$, $s_{AB}\geq 0$ and $0\leq\delta_{13}\leq 2\pi$. In the limit $\theta_{13}=\theta_{23}=0$, the third generation decouples, and the situation reduces to the Cabibbo mixing of the first two generations, with θ_{12} identified with θ_{C}, the Cabibbo angle.

9.4.4 Summary and Extensions

In the original gauge-invariant Lagrangian, all fields behave as eigenstates of the gauge group. In particular, the fermion fields display a repetitive pattern and may be grouped into generations, each generation composed of a doublet of left-handed leptons, a doublet of left-handed quarks, a right-handed charged lepton, and two right-handed quarks. It is assumed that all neutrinos are massless. For three generations, $N_{\mathrm{G}}=3$, we have 21 chiral fields in all. General gauge-invariant Yukawa interactions couple generations together and lead to nondiagonal mass matrices, one matrix for each set of equal charge fermions. When the quark mass matrices are diagonalized, the gauge eigenstates of the d-type quarks appear as linear combinations of the mass eigenstates via unitary transformations. As for the leptons, the mass eigenstates are unmixed because of the assumed mass degeneracy (that is, complete absence of mass) of the neutrinos.

The final form of the Lagrangian of the standard model is written in the mass eigenstates. The gauge field and scalar field sectors are given in (112) and (115). The fermion sector, studied in this section, appears grouped into three families, differing from one another by their masses and flavor quantum numbers (see Table 9.5) but having essentially identical electroweak interaction properties. Their dynamics is described by the following terms:

Free-quark fields:

$$\begin{aligned}\mathcal{L}_{\mathrm{F}}^{0} &= \overline{\nu}_A\,\mathrm{i}\gamma^{\mu}\partial_{\mu}\,a_{\mathrm{L}}\nu_A+\overline{e}_A\,\left(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m_A^{\mathrm{e}}\right)\,e_A \\ &\quad +\overline{u}_A\,\left(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m_A^{\mathrm{u}}\right)\,u_A+\overline{d}_A\,\left(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m_A^{\mathrm{d}}\right)\,d_A\,; \end{aligned} \quad (9.179)$$

Table 9.5. Family pattern of the fundamental fermions in the standard model after spontaneous symmetry breaking, with d'', s'', and b'' denoting orthogonal combinations of d, s, and b defined by the CKM matrix, $d''_A = V_{AB} d_B$

1	2	3	Q
$\begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{L}} \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu\mathrm{L}} \\ \mu_{\mathrm{L}} \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau\mathrm{L}} \\ \tau_{\mathrm{L}} \end{pmatrix}$	$\begin{matrix} 0 \\ -1 \end{matrix}$
e_{R}	μ_{R}	τ_{R}	-1
$\begin{pmatrix} u_{\mathrm{L}} \\ d''_{\mathrm{L}} \end{pmatrix}$	$\begin{pmatrix} c_{\mathrm{L}} \\ s''_{\mathrm{L}} \end{pmatrix}$	$\begin{pmatrix} t_{\mathrm{L}} \\ b''_{\mathrm{L}} \end{pmatrix}$	$\begin{matrix} \frac{2}{3} \\ -\frac{1}{3} \end{matrix}$
u_{R}	c_{R}	t_{R}	$\frac{2}{3}$
d_{R}	s_{R}	b_{R}	$-\frac{1}{3}$

Higgs couplings:

$$\mathcal{L}_{\mathrm{FH}} = -\frac{g}{2\,M_{\mathrm{W}}} \left(m^{\mathrm{e}}_A \, \overline{e}_A e_A \, + m^{\mathrm{u}}_A \, \overline{u}_A u_A + m^{\mathrm{d}}_A \, \overline{d}_A d_A \right) \, H \, ; \qquad (9.180)$$

Electromagnetic coupling:

$$\mathcal{L}_{\mathrm{em}} = -\, e \big(Q_{\mathrm{e}} \, \overline{e}_A \gamma_\mu e_A + Q_{\mathrm{u}} \, \overline{u}_A \gamma_\mu u_A + Q_{\mathrm{d}} \, \overline{d}_A \gamma_\mu d_A \big) \, A^\mu \, ; \qquad (9.181)$$

Neutral current coupling:

$$\begin{aligned} \mathcal{L}_{\mathrm{nc}} = -\, \frac{g}{2\,c_{\mathrm{W}}} \big[& \overline{\nu}_A \gamma_\mu (g^\nu_{\mathrm{V}} - g^\nu_{\mathrm{A}} \gamma_5) \nu_A + \overline{e}_A \gamma_\mu (g^{\mathrm{e}}_{\mathrm{V}} - g^{\mathrm{e}}_{\mathrm{A}} \gamma_5) \, e_A \\ & + \overline{u}_A \gamma_\mu (g^{\mathrm{u}}_{\mathrm{V}} - g^{\mathrm{u}}_{\mathrm{A}} \gamma_5) \, u_A + \overline{d}_A \gamma_\mu (g^{\mathrm{d}}_{\mathrm{V}} - g^{\mathrm{d}}_{\mathrm{A}} \gamma_5) \, d_A \big] \, Z^\mu \, ; \end{aligned} \qquad (9.182)$$

Charged current coupling:

$$\mathcal{L}_{\mathrm{cc}} = -\, \frac{g}{\sqrt{2}} \left[\big(\overline{\nu}_A \, \gamma_\mu \, a_{\mathrm{L}} \, e_A + \overline{u}_A \, \gamma_\mu \, a_{\mathrm{L}} \, V_{AB} \, d_B \big) \, W^\mu + \text{ h.c.} \right] , \qquad (9.183)$$

where $a_{\mathrm{L}} = \frac{1}{2} \, (1 - \gamma_5)$.

The weak charged currents display the required V–A structure, a well-established fact of low-energy physics. The W bosons couple with the same strength, $g/2\sqrt{2}$, to all charged fermionic currents (up to CKM mixing factors in the quark sector). The weak neutral currents, a new feature introduced by the unified model, conserve flavor and display universality in interaction: they couple to the Z field with the same coupling strengths in all generations, the values of $g^{\mathrm{f}}_{\mathrm{V}}$ and $g^{\mathrm{f}}_{\mathrm{A}}$ given in Table 9.3 for the first generation being also valid in general for every lepton or quark having the indicated charge.

In the gauge and scalar sectors, the model contains four parameters, g, g', λ, and μ^2, or alternatively, e, θ_{W}, M_{W}, and M_{H}. In the fermion sector, with $N_{\mathrm{G}} = 3$, thirteen parameters are needed: three charged lepton masses and six quark masses plus three quark mixing angles and one phase, all originating from the unknown Yukawa couplings $\boldsymbol{C}^f$.

The model can be readily extended to include the *strong force* treated as the gauge interaction based on the color SU(3) group. As leptons are insensitive to this force, they are regarded as singlets under $\mathrm{SU_c(3)}$ (so also are the Higgs fields), but quarks belong to the fundamental triplet representations. Evidently, as the generators of the color group commute with the weak isospin and the weak hypercharge, the group of symmetry to be gauged is the direct product group

$$\mathrm{SU_c(3)} \times \mathrm{SU_L(2)} \times \mathrm{U_Y(1)}\,. \tag{9.184}$$

It is assumed that $\mathrm{SU_c(3)}$ remains unbroken, whereas $\mathrm{SU_L(2)}\times\mathrm{U_Y(1)}$ spontaneously breaks down to $\mathrm{U_Q}$. This symmetry breaking is represented schematically by

$$\mathrm{SU_c(3)} \times \mathrm{SU_L(2)} \times \mathrm{U_Y(1)} \rightarrow \mathrm{SU_c(3)} \times \mathrm{U_Q(1)}\,. \tag{9.185}$$

Since (184) is a direct product group, no complications arise in the formulation, but neither can any relationships between the strong and the electroweak forces emerge from this juxtaposition of two gauge theories. The resulting Lagrangian for the gauge group $\mathrm{SU_c(3)} \times \mathrm{SU_L(2)} \times \mathrm{U_Y(1)}$ is essentially just the sum of (8.46), (112)–(115), and (179)–(183). The number of parameters has now increased by one, adding the strong coupling constant g_s, for a total of 18 in the case of three complete generations of fermions.

Thus, we now have the basic elements of a theory that proves to be consistent with the present state of our knowledge of particle physics and that in some cases (e.g. in weak neutral current processes) can pass stringent experimental tests at a very high degree of precision. In the following chapters, we shall re-examine the underlying assumptions of the theory, study a number of predictions, and introduce further theoretical concepts essential for a more complete theory.

Among the ingredients of the standard model, none is more important than the 'elementarity' of quarks. The evidence for this key property, as found in deep inelastic electron–nucleon and neutrino–nucleon scattering, will be considered in Chaps. 10 and 12. The latter chapter also dwells on the assumption of massless neutrinos (and hence that of the conservation of the lepton numbers), while Chap. 13 presents further proof of the universality of the left-handed structure of the weak charged current.

As mentioned above, the gauge sector of the electroweak theory contains three independent parameters, a convenient choice of which is

$$\begin{cases} \dfrac{4\pi}{e^2} = \alpha^{-1} = 137.0359895 \pm 0.0000061\,, \\ G_\mathrm{F} = (1.16639 \pm 0.00002) \times 10^{-5}\,\mathrm{GeV}^{-2}\,, \\ M_\mathrm{Z} = (91.1888 \pm 0.0044)\,\mathrm{GeV}\,. \end{cases} \tag{9.186}$$

The fine structure constant α can be determined from the quantum Hall effect, the Fermi constant G_F from the muon lifetime formula, and the Z^0

gauge boson mass M_Z from e^+e^- and $p\bar{p}$ collider experiments. The model makes several very definite predictions, the simplest being on the charged boson mass and the weak mixing angle:

$$M_W^2 s_W^2 = \pi\alpha/(\sqrt{2}\,G_F)\,, \qquad s_W^2 = 1 - M_W^2/M_Z^2\,. \tag{9.187}$$

These relations, which follow from $e = gs_W$, $G_F/\sqrt{2} = g^2/(8M_W^2)$, and $M_W^2 = M_Z^2 c_W^2$, yield $M_W \approx 80.94$ GeV and $s_W^2 \approx 0.212$. The small differences with the measured values ($M_W = 80.33\,\text{GeV}$ and $s_W^2 = 0.2315$) are attributable to higher-order quantum corrections. Similar corrections, which turn out to be more substantial in several important observables, will be the subject of discussion in Chaps. 14 and 15. (The latter contains also a detailed study of essential properties of QCD).

Whereas the weak *neutral* current is well known, as it depends only on α and θ_W, it is not the case of the weak *charged* current for quarks since its coupling constant depends also on the CKM parameters. For the lighter quarks, the magnitudes of the matrix elements V_{AB} can be evaluated directly from the rates of the quark transitions $q_A \to q_B \ell^- \bar{\nu}_\ell$. Since quarks are confined, the relevant processes are the corresponding leptonic weak decays of hadrons $H \to H' \ell^- \bar{\nu}_\ell$; their amplitudes always involve hadronic matrix elements of the weak charged currents – a nonperturbative QCD problem. For those matrix elements involving the b quark, the heavy-flavor symmetry allows a clean calculation of the hadronic form factor, but for those relating to the top quark, they can be accessed only indirectly, e.g. through the top's participation in the B^0–$\overline{B}^0$ mixing. Considerations of this kind are found in Chap. 16. The Kobayashi–Maskawa phase is even harder to come by; the best that can be done is to subject its value to constraints derived from the CP violation parameters ε and ε' of the neutral K mesons and from the K^0–$\overline{K}^0$ and B^0–$\overline{B}^0$ mixings (Chaps. 11 and 16).

Finally, the mass of the physical neutral Higgs scalar is not predicted by the model. It remains the most poorly known parameter in the model, and the existence and real nature of the Higgs boson is the object of an active research (Chap. 17).

Problems

9.1 Necessity of conserved current. Consider the coupling of a vector field of mass M to a current of the form $j_\mu(x)A^\mu(x)$. Assume that in momentum space the vector field satisfies the Lorentz condition $k_\mu A^\mu = 0$, for any particle momentum k_μ, to give three independent physical components. For a nonvanishing mass M, decompose A^μ into a transverse part $A_\perp^\mu$ (defined by $k_\mu A_\perp^\mu = 0$ and $\boldsymbol{k}\cdot\boldsymbol{A} = 0$) and its orthogonal longitudinal complement, $A_\parallel^\mu$. Let the first-order transition matrix be $\mathcal{M} = T_\mu A^\mu$, where

$T_\mu = \langle \mathrm{f} \,|\, j_\mu \,|\, \mathrm{i} \rangle$. Show that the longitudinal vector $A^\mu_\parallel$ increases with energy and causes ultraviolet divergences in $\mathcal{M}$ unless the current is conserved.

9.2 Massive neutrino. In the one-lepton family, make the appropriate modifications when the neutrino is massive, and justify the neglect of the right-handed component of the neutrino field in the limit of zero mass.

9.3 Numerical estimates of parameters of the model. It is convenient to take as inputs to the model the three parameters in (186) plus M_H and the fermion masses. Define also $A = \pi\alpha/\sqrt{2}G_\mathrm{F} = M_\mathrm{W}^2 s_\mathrm{W}^2$. From these data, calculate M_W, θ_W, and v. In addition, with $m_\mathrm{e} = 0.511$ MeV, calculate the Yukawa coupling constant C_e. Note that since $M_\mathrm{H}^2 = 2\lambda v^2$ and since there is no simple way of obtaining λ, it is not possible to predict its value.

9.4 Decay width of $\mathbf{W}^\pm$. Assuming the electron is massless and the W–lepton coupling given in the form $-(g/2\sqrt{2})\bar{e}\gamma^\mu(1-\gamma_5)\nu\, W_\mu^\dagger + \text{h.c.}$, calculate the decay width $\Gamma(\mathrm{W}^+ \to \mathrm{e}^+\nu)$. Assuming the quarks are also massless, calculate to lowest order the decay widths of W^+ to various allowed quark–antiquark channels, and give an estimate of the total decay width of W.

9.5 Decay width of $\mathbf{Z}^0$. The coupling of the Z boson to fermions is given by $(-g/c_W) j_\mu^Z\, Z^\mu$, where j_μ^Z is the weak neutral current for fermions. Calculate to lowest order the decay width $\Gamma(\mathrm{Z}^0 \to \nu\bar{\nu})$ and compare it with $\Gamma(\mathrm{W}^+ \to \mathrm{e}^+\nu)$. Also calculate to lowest order the rates of decay of Z to various allowed lepton–antilepton and quark–antiquark channels. Give an estimate of the total decay width of Z.

9.6 Front–back asymmetry for $\mathrm{e}^+ + \mathrm{e}^- \to \mathrm{f}^+ + \mathrm{f}^-$**.** In the reaction $\mathrm{e}^-(p) + \mathrm{e}^+(p') \to \mathrm{f}^-(k) + \mathrm{f}^+(k')$, where f is a charged fermion, there is a relative difference between the probabilities of observing f^- traveling in the forward (σ_F) and backward (σ_B) directions due to an interference between the contributions of the photon γ and the weak boson Z^0 exchanged in the s-channel. Compute the asymmetry of the total cross-sections, $A_\mathrm{FB} = (\sigma_\mathrm{F} - \sigma_\mathrm{B})/(\sigma_\mathrm{F} + \sigma_\mathrm{B})$. A measure of this quantity would give the Weinberg angle.

Suggestions for Further Reading

The classic papers:

Glashow, S. L., Nucl. Phys. **22** (1961) 579

Salam, A., in *Elementary Particle Theory* (ed. by N. Svartholm). Almquist and Wiksells, Stockholm 1968, p. 367

Weinberg, S., Phys. Rev. Lett. **19** (1967) 1264

Introduction of the charmed quark:

Glashow, S. L., Iliopoulos, J. and Maiani, L., Phys. Rev. **D2** (1970) 1285

Quark mixing matrix:

Cabibbo, N., Phys. Rev. Lett. **10** (1963) 531

Kobayashi, M. and Maskawa, T., Progr. Theor. Phys. **49** (1973) 652