# 8 Gauge Field Theories

All the current successful theories of the fundamental forces start from the premise of invariance of the physical laws to certain coordinate-dependent transformations. In particular, the quantum field theories of the electromagnetic, weak, and strong interactions of the fundamental particles all belong to the class of *local gauge theories*, so called because they are invariant to coordinate-dependent transformations on internal space of the particles. We start this chapter by describing the general relation between symmetries and interactions. Next, we take up the study of invariance under the Abelian gauge group U(1), the group of space-time-dependent phase transformations on charged fields; the resulting gauge theory is electrodynamics. The following section is devoted to theories for which the gauge group is non-Abelian. The results see immediate applications to quantum chromodynamics, a theory based on the color SU(3) group. The last two sections of the chapter contain a discussion on the mechanism of *spontaneous symmetry breaking*, which is an indispensable ingredient in the formulation of the standard theory of the electroweak interaction, the subject of the following chapter.

## 8.1 Symmetries and Interactions

In previous chapters, we have studied some of the implications of the conservation or violation of *global* symmetries that a theory may have. Under a symmetry transformation of this kind, fields are changed by an identical amount that remains fixed throughout space and time, and invariance of the theory to such changes implies the existence of a conserved quantity.

Generally, when this symmetry is made *local*, whereby all the particle fields are altered by an amount that varies with each space-time point, invariance may be preserved provided a set of vector fields (or higher-rank tensor fields) defined over all space-time is introduced into the theory to cancel the long-range effects of the vector gradient of the transformation parameter and to restore the symmetry. In particular, transformations in which this parameter is the phase-angle of the particle fields are called *gauge transformations* and considered as *internal*, for they act on the labels of the particles rather than on their space-time coordinates. If global, they give rise to conserved charges, of which the electric charge is an example. If local, they may lead to an observable force. Three of the four existing fundamental interactions are believed to be explicable in this fashion: the electric current is the source of the electromagnetic force; the weak isospin and the weak hypercharge are the sources of the unified electromagnetic and weak forces; and finally, the quark colors, the sources of the strong interactions between quarks. It is not understood why only those and no other conserved charges can produce observable dynamical effects. As for the fourth fundamental force, gravitation, it may be similarly viewed as arising from a local invariance, with the difference that the transformations that leave the theory invariant act on space-time coordinates themselves, and therefore the resulting force field, generated by the conserved energy-momentum tensor, is tensorial rather than vectorial. Otherwise, gravitation and gauge theories have close similarities with one another.

In any nontrivial quantum field theory, divergent integrals may appear in higher orders of the perturbation expansion of the transition amplitudes. Renormalization is a procedure of removing these ultraviolet divergences by adding extra terms, called counterterms, to the Lagrangian of the theory. A theory is said to be *renormalizable* when all the counterterms induced by this procedure are of the same form as terms in the original Lagrangian. A theory with an interaction of mass dimension greater than four is nonrenormalizable, although not all theories involving only interactions of mass dimension four or less are necessarily renormalizable. All three gauge theories mentioned above respect this simple but highly constraining demand of renormalizability. We shall return to this topic in Chap. 15.

As mentioned above, the vector gauge fields introduced into the theory to enforce gauge invariance have an infinite range or, equivalently, have no mass. The photon, which is supposed to mediate the electromagnetic interaction, is in effect observed to be massless ($m_\gamma < 6 \times 10^{-16}$ eV). Experiment is also consistent with the assumption that the gluons, the gauge fields of the fundamental strong force, have vanishing masses. However, the weak forces have always been known since their discovery to have a very short range. So the gauge fields associated with the conserved weak charges in a gauge-invariant theory cannot be immediately identified with the observed weak forces. They must first acquire mass. But we are not allowed to introduce artificially a mass term in the theory since this would break gauge invariance, which would in turn make the theory divergent and thus nonpredictive. The solution to this difficulty is to hide part of the gauge group, that is, to arrange so that, while remaining exact in the underlying field equations, the gauge symmetries are not realized in physical states. This *spontaneous symmetry breakdown* is similar to the loss of symmetry during certain phase transitions observed in condensed matter, such as the loss of translational symmetry when liquid water turns into an ice crystal lattice below $0^\circ$ C, or the loss of rotational symmetry when a very large sample of ferromagnet acquires a net magnetization below the Curie temperature.

## 8.2 Abelian Gauge Invariance

In quantum electrodynamics (QED), the interaction of a charged particle with an electromagnetic field is obtained by coupling the field with the electromagnetic current for the particle, an empirical rule known in classical physics as the *minimal coupling postulate*. This rule can be better understood in terms of a general symmetry principle, called the *principle of gauge invariance*, susceptible of generalizations.

Consider, as a representative example of matter, a fermion field. The Lagrangian density for a free Dirac field of mass $m$ is

$$\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi \,. \tag{8.1}$$

It is invariant to the *global* phase transformation

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega}\, \psi(x)\,, \tag{8.2}$$

where $\omega$ is the transformation parameter, an arbitrary constant number, independent of $x$. Another constant $q$ has been inserted at this point to accord with common usage; it will take on the meaning of the particle electric charge in the present context. All operations (2) form a representation of the single-parameter Abelian group $\mathrm{U}(1)$, which is sometimes written with a subscript, such as in $\mathrm{U_Q}(1)$, to emphasize its association with a conserved quantum number. It is crucial for the invariance of $\mathcal{L}_0(\psi, \partial_\mu\psi)$ to the global symmetry transformation (2) that the field gradient transforms exactly like the field itself:

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega}\, \partial_\mu\psi(x)\,. \tag{8.3}$$

As discussed in Sect. 3.4, this invariance implies the existence of a locally conserved current,

$$j^\mu(x) = q\,\overline{\psi}\gamma^\mu\psi\,. \tag{8.4}$$

The global transformation (2) can be generalized to the *local* transformation

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega(x)}\, \psi(x)\,, \tag{8.5}$$

where $\omega$ is now a real function of $x$, i.e. $\omega(x)$ defines an independent phase transformation at each space-time point. However, the Lagrangian (1) and, in general, any free-field Lagrangian cannot be invariant under this local transformation because the transformation rule for the field gradient differs from that for the field,

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = \mathrm{e}^{-\mathrm{i}q\omega}\, [\partial_\mu\psi - \mathrm{i}q(\partial_\mu\omega)\,\psi\,]\,, \tag{8.6}$$

so that the transformed free-field Lagrangian acquires an extra term which spoils the invariance:

$$\mathcal{L}_0 \to \mathcal{L}_0' = \mathcal{L}_0 + q\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\omega = \mathcal{L}_0 + j^\mu\partial_\mu\omega\,. \tag{8.7}$$

The presence of a symmetry-violating term in (7) suggests that if we wish to make the theory invariant under (5), it is necessary to introduce a vector field $A_\mu$ that couples to the particle current so that this coupling when transformed may cancel $j^\mu\partial_\mu\omega$. The modified Lagrangian

$$\mathcal{L}_1 = \mathcal{L}_0 - j^\mu A_\mu \tag{8.8}$$

transforms under (5) as

$$\mathcal{L}_1 \to \mathcal{L}_1' = \mathcal{L}_0' - j'^\mu A_\mu' = \mathcal{L}_0 + j^\mu\,\partial_\mu\omega - j^\mu\,A_\mu' \tag{8.9}$$

(since $j_\mu' = q\,\overline{\psi}'\gamma_\mu\psi' = j_\mu$). Invariance of $\mathcal{L}_1$ then requires the vector field to have the property

$$A_\mu \to A_\mu' = A_\mu + \partial_\mu\omega \tag{8.10}$$

under the transformation that acts on $\psi$ according to (5). The 'scale' of the vector field has changed. Thus, the quantum theory of electrically charged particles is said to have local phase-angle independence – referring to the change of matter field in (5) – or more currently, local gauge invariance – emphasizing the scale change of the force field in (10). The field $A_\mu$ is accordingly called a *gauge field*. Rewriting the Lagrangian as

$$\mathcal{L}_1 = \overline{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi - q\,\overline{\psi}\gamma^\mu\psi A_\mu \;= \overline{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi\,, \tag{8.11}$$

where

$$D_\mu \equiv \partial_\mu + \mathrm{i}qA_\mu\,, \tag{8.12}$$

we observe that the gauge invariance of $\mathcal{L}_1$ has been realized by making the field gradient transform covariantly, that is,

$$D_\mu\psi \to D_\mu'\psi' = \mathrm{e}^{\mathrm{i}q\omega}\,D_\mu\psi\,, \tag{8.13}$$

and, for this reason, $D_\mu$ is called a *covariant derivative* for this gauge group. To make the vector field an integral part of the dynamic system, it is necessary to introduce gauge-invariant terms built up from $A_\mu$ and its derivatives. The combination

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under (10), whereas

$$\partial_\mu A_\nu + \partial_\nu A_\mu$$

is not. Therefore, the Lorentz scalar $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (with a conventional normalization factor) may be added to the Lagrangian. The mass term $A_\mu A^\mu$ is not allowed since it is not invariant under (10). So the final gauge-invariant Lagrangian looks like

$$\mathcal{L}_1 = \overline{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{8.14}$$

The term $\epsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}F_{\rho\sigma}$, which is equally gauge-invariant and of dimension four, need not be included because it may be rewritten as the divergence of a current, $\partial_\mu K^\mu$, and therefore contributes only as a surface term to the action. Under the usual assumption that fields vanish at infinity, it may be discarded. Other higher-dimensional gauge-invariant couplings, such as $\overline{\psi}\sigma_{\mu\nu}\psi\,F^{\mu\nu}$, are not allowed by the requirement of renormalizability. When the fields that appear here are reinterpreted as quantum fields, (14) is just the familiar form of the QED Lagrangian for a Dirac particle of charge $q$ interacting with the electromagnetic field. It is the most general U(1)-gauge-invariant dimension-four renormalizable Lagrangian, and it is in extremely good agreement with experiment.

Thus, we have shown that, when a free-field theory has an exact global phase symmetry, it may have the corresponding local phase invariance only upon becoming an interacting field theory involving a massless vector field (the photon) which interacts with the charged particle in a well-defined manner. In Abelian theories such as this one, there are no restrictions on the coupling strength between the gauge field and matter fields; the electron has charge $q = -e$ while another particle may carry any other charge $q = ze$. But the interaction appears in the same form regardless of the nature of the particle, be it lepton, quark, or hadron. That the interaction derived from imposing renormalizability and some kind of gauge invariance on the theory is *unique* and *universal* is precisely what has made this symmetry condition – the *gauge invariance principle* – so powerful that it has now become the guiding principle in the search for the theories of interactions in particle physics.

## 8.3 Non-Abelian Gauge Invariance

As particles usually come in multiplets, we might wonder what kind of gauge fields and interactions the principle of gauge invariance would imply in general. Suppose, for example, we have a number of Dirac spinor fields $\psi^a$ (with $a = 1, \dots, n$ describing some internal degree of freedom, such as isospin or color) that form such a multiplet, $\boldsymbol{\psi}$; that is to say, they have equal masses, $m_a = m$, and transform into one another by the rule

$$\psi^a \to \psi'^a = U^a{}_b\,\psi^b\,, \tag{8.15}$$

where $U$ is a *unitary* $n \times n$ matrix. In the following, we will further limit ourselves to *unimodular* matrices, so that $\det U = 1$. All such matrices define some representation of a Lie group, G. To simplify, we also assume that G is a simple group and $\boldsymbol{\psi}$ belongs to its fundamental representation.

Unitary unimodular matrices $U$ may be parameterized by $N = n^2 - 1$ real phase-angles $\omega_i$ in the form

$$U = \exp(-\mathrm{i}gT_i\omega_i)\,, \tag{8.16}$$

where, as usual, a sum over repeated indices is implied. The real constant factor $g$, common to all terms in the sum, will turn out to be a coupling constant. Transformations very near the identity are given by $1 - \mathrm{i}gT_i\omega_i$, and for this reason the matrices $T_i$ are called the generators of infinitesimal transformations. They are Hermitian and traceless, $T_i^\dagger = T_i$ and $\mathrm{Tr}\, T_i = 0$, as respective consequences of the unitarity and unimodularity of $U$. They constitute a basis of a Lie algebra, and must satisfy commutation relations of the form

$$[T_i,\, T_j] = \mathrm{i}f_{ijk}\, T_k\,, \qquad \text{for} \quad i,\, j,\, k = 1, \ldots, N\,. \tag{8.17}$$

When not all the structure constants $f_{ijk}$ vanish, these relations define a *non-Abelian* algebra. It is convenient to normalize the generators such that

$$\mathrm{Tr}\ (T_iT_j) = \tfrac{1}{2}\,\delta_{ij}\,. \tag{8.18}$$

For G=SU(2), the generators in the fundamental representation are given by the familiar $2 \times 2$ Pauli matrices, $T_i = \frac{1}{2}\,\tau_i$, with $i = 1, 2, 3$, while for G=SU(3), $T_i = \frac{1}{2}\,\lambda_i$, with $i = 1, \ldots, 8$, are the $3 \times 3$ Gell-Mann matrices.

The free-field Lagrangian, assumed independent of the internal degree of freedom, is given by

$$\mathcal{L}_0 = \overline{\psi}_a\,(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi^a \; = \overline{\boldsymbol{\psi}}\;(\mathrm{i}\gamma^\mu\partial_\mu - m)\;\boldsymbol{\psi}\,. \tag{8.19}$$

In the second equation, the operator contains an implicit unit matrix defined on the $n$-dimensional space of the group representation.

The free-field Lagrangian is invariant under the gauge transformation (15) provided it is a *global* transformation, independent of space-time coordinates $x$. The conserved fermion currents that follow from this invariance are given by Noether's theorem:

$$j_i^\mu = g\,\overline{\boldsymbol{\psi}}\,\gamma^\mu T_i\,\boldsymbol{\psi}\,. \tag{8.20}$$

For the isospin group SU(2), they are the conserved isospin currents.

On the other hand, if $U$ represents a *local* transformation, which depends on the space-time point where it acts, $U = U(x)$, then the free-field Lagrangian will not be invariant in general, but will rather vary as

$$\mathcal{L}_0 \to \mathcal{L}_0' = \mathcal{L}_0 + \overline{\boldsymbol{\psi}}\,\mathrm{i}\gamma^\mu(U^\dagger\partial_\mu U)\,\boldsymbol{\psi}\,, \tag{8.21}$$

where the symmetry-violating term arises from differences in the transformation rules for the field and the field gradient:

$$\boldsymbol{\psi} \to \boldsymbol{\psi}' = U\boldsymbol{\psi}, \tag{8.22}$$

$$\partial_\mu \boldsymbol{\psi} \to \partial_\mu \boldsymbol{\psi}' = U\,\partial_\mu \boldsymbol{\psi} + (\partial_\mu U)\boldsymbol{\psi}\,. \tag{8.23}$$

This suggests that we must introduce extra fields with couplings to the Noether currents similar in form to the second term on the right-hand side of (21) to compensate for this unwanted term. The modified Lagrangian

$$\mathcal{L}_1 = \mathcal{L}_0 - g\overline{\boldsymbol{\psi}}\gamma^\mu \boldsymbol{A}_\mu\, \boldsymbol{\psi}\,, \tag{8.24}$$

where $\boldsymbol{A}_\mu$ is an $n \times n$ Hermitian traceless matrix whose elements are vector fields, transforms as

$$\begin{aligned}\mathcal{L}_1 \to \mathcal{L}_1' &= \mathcal{L}_0' - g\overline{\boldsymbol{\psi}}'\gamma^\mu \boldsymbol{A}_\mu'\, \boldsymbol{\psi}' \\ &= \mathcal{L}_0 + \overline{\boldsymbol{\psi}}\,\mathrm{i}\gamma^\mu (U^\dagger \partial_\mu U)\boldsymbol{\psi} - g\overline{\boldsymbol{\psi}}\,\gamma^\mu\, U^\dagger \boldsymbol{A}_\mu' U\,\boldsymbol{\psi}\,.\end{aligned} \tag{8.25}$$

The demand that $\mathcal{L}_1$ be invariant in this operation requires

$$\overline{\boldsymbol{\psi}}\,\mathrm{i}\gamma^\mu (U^\dagger \partial_\mu U)\boldsymbol{\psi} - g\overline{\boldsymbol{\psi}}\gamma^\mu\, U^\dagger \boldsymbol{A}_\mu' U\,\boldsymbol{\psi} = -g\overline{\boldsymbol{\psi}}\gamma^\mu \boldsymbol{A}_\mu\, \boldsymbol{\psi}\,.$$

Thus, in a gauge transformation that acts on $\boldsymbol{\psi}$ according to (15), the vector field has the transformation property

$$\boldsymbol{A}_\mu \to \boldsymbol{A}_\mu' = \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger + U\boldsymbol{A}_\mu U^\dagger\,. \tag{8.26}$$

For most practical purposes it suffices to restrict $\omega_i(x)$ to infinitesimal values so that, to first order,

$$U \approx 1 - \mathrm{i}g\,\boldsymbol{\omega}\,, \tag{8.27}$$

where $\boldsymbol{\omega} = \omega_j T_j$ . To this order, the fermion field transforms as

$$\begin{aligned}\boldsymbol{\psi}' &= U\boldsymbol{\psi} \approx \boldsymbol{\psi} + \delta\boldsymbol{\psi}\,, \\ \delta\boldsymbol{\psi} &= -\,\mathrm{i}g\,\boldsymbol{\omega}\,\boldsymbol{\psi}\,,\end{aligned} \tag{8.28}$$

or in components

$$\delta\psi^a = -\mathrm{i}g\,\omega_j\,(T_j)^a{}_b\,\psi^b \qquad \text{for } a = 1, \ldots, n\,. \tag{8.29}$$

There are $(n^2 - 1)$ local gauge fields $A_{j\mu}$, which are independent of the representation of the particle fields $\boldsymbol{\psi}$ and which form the elements of the matrices $\boldsymbol{A}_\mu$. They are chosen so that $\boldsymbol{A}_\mu = A_{j\mu} T_j$. To first order, the transformation rule (26) for the gauge field matrix becomes

$$\begin{aligned} \boldsymbol{A}'_\mu &= \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger + U\boldsymbol{A}_\mu U^\dagger \approx \boldsymbol{A}_\mu + \delta\boldsymbol{A}_\mu\,, \\ \delta\boldsymbol{A}_\mu &= \partial_\mu \boldsymbol{\omega} + \mathrm{i}g\,[\boldsymbol{A}_\mu, \boldsymbol{\omega}\,]\,, \end{aligned} \tag{8.30}$$

or in components

$$\delta A_{i\mu} = \partial_\mu \omega_i - g\, f_{ijk}\, A_{j\mu}\, \omega_k \qquad \text{for } i = 1, \ldots, n^2 - 1\,. \tag{8.31}$$

If G is Abelian, (31) reduces to

$$\delta A_{i\mu} = \partial_\mu \omega_i \qquad \text{(Abelian group)}\,.$$

It corresponds to the first, inhomogeneous term in (31) and implies that the vector fields $A_{i\mu}$ have as sources the currents $j_i^\mu$, just as the transformation rule for the electromagnetic field identifies the electric current as its source. If, on the other hand, G is a non-Abelian group of global symmetry, the transformation rule becomes

$$\delta A_{i\mu} = -g\, f_{ijk}\, A_{j\mu}\, \omega_k \qquad \text{(global symmetry)}\,.$$

The right-hand side of this equation is the same in form as the right-hand side of (29) with $(T_j)^a{}_b$ replaced by $-\mathrm{i}f_{jab}$, which indicates that the gauge fields $A_{i\mu}$ belong to the adjoint representation of the group; that is, for example, they transform as an isovector in SU(2) and as an octet in SU(3).

In terms of the covariant derivative for the non-Abelian gauge group G

$$\boldsymbol{D}_\mu = \partial_\mu + \mathrm{i}g\boldsymbol{A}_\mu\,, \tag{8.32}$$

which obeys the relation

$$\boldsymbol{D}'_\mu U\boldsymbol{\psi} = U\,\boldsymbol{D}_\mu \boldsymbol{\psi}\,, \tag{8.33}$$

the Lagrangian (24) takes the form

$$\mathcal{L}_1 = \overline{\boldsymbol{\psi}}\,\mathrm{i}\gamma^\mu\,(\partial_\mu + \mathrm{i}g\boldsymbol{A}_\mu)\boldsymbol{\psi} - m\,\overline{\boldsymbol{\psi}}\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}\,(\mathrm{i}\gamma^\mu \boldsymbol{D}_\mu - m)\,\boldsymbol{\psi}\,. \tag{8.34}$$

To this Lagrangian must be added contributions from the gauge fields themselves. In analogy with the identity

$$[D_\mu,\, D_\nu\,]\,\psi = \mathrm{i}q\,F_{\mu\nu}\psi\,, \tag{8.35}$$

which is satisfied by the electromagnetic field strength, we may define the field tensor in the non-Abelian case by the generalized relation

$$[\boldsymbol{D}_\mu,\, \boldsymbol{D}_\nu\,]\,\boldsymbol{\psi} \equiv \mathrm{i}g\,\boldsymbol{F}_{\mu\nu}\boldsymbol{\psi}\,. \tag{8.36}$$

Under the gauge transformation $U$, the left-hand side of (36) gives

$$[\boldsymbol{D}'_\mu,\ \boldsymbol{D}'_\nu]\, U\boldsymbol{\psi} = U[\boldsymbol{D}_\mu,\ \boldsymbol{D}_\nu]\,\boldsymbol{\psi} = \mathrm{i}g\, U\boldsymbol{F}_{\mu\nu}\boldsymbol{\psi}\,,$$

where (33) has been used, while the right-hand side becomes

$$\mathrm{i}g\,\boldsymbol{F}'_{\mu\nu}\,\boldsymbol{\psi}\,' = \mathrm{i}g\,\boldsymbol{F}'_{\mu\nu}\,U\boldsymbol{\psi}\,.$$

Therefore, identifying the right-hand sides of the last two equations yields

$$\boldsymbol{F}'_{\mu\nu} = U\boldsymbol{F}_{\mu\nu}U^\dagger \approx \boldsymbol{F}_{\mu\nu} + \mathrm{i}g\,[\boldsymbol{F}_{\mu\nu},\,\boldsymbol{\omega}\,]\,. \tag{8.37}$$

Thus, we have learned that the non-Abelian field strength is not invariant, merely covariant; it transforms under (26) as an adjoint multiplet, just like $\boldsymbol{A}_\mu$ but without an inhomogeneous term. Since $\boldsymbol{F}_{\mu\nu}$ is an $n\times n$ matrix, it decomposes as

$$\boldsymbol{F}_{\mu\nu} = F^i_{\mu\nu}\,T_i\,, \tag{8.38}$$

where the expression

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\,f_{ijk}\,A^j_\mu A^k_\nu \tag{8.39}$$

shows that the field strengths are independent of the fermion representation chosen in the defining relation (36). The kinetic term in the electromagnetic Lagrangian admits as non-Abelian generalization

$$\mathrm{Tr}\,(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu})\,, \tag{8.40}$$

which is both Lorentz- and gauge-invariant, as it should be. With the help of the orthonormality relation (18) for $T_i$, it may also be rewritten as

$$\mathrm{Tr}\,(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = F^i_{\mu\nu}F^{\mu\nu}_j\,\mathrm{Tr}(T_iT_j) = \tfrac{1}{2}\,F^i_{\mu\nu}F^{\mu\nu}_i\,. \tag{8.41}$$

To summarize, the free-field Lagrangian (19) is invariant in the global non-Abelian symmetry group (15), but not in the corresponding local gauge group. Application of the principle of gauge invariance turns it into an interacting field theory when one introduces vector gauge fields, as many fields as there are generators in the gauge group and appropriately coupled to the conserved vector currents (20). The first theory of this type [for the case of the isospin SU(2) group] was constructed by C. N. Yang and R. L. Mills; for this reason a theory invariant under a local non-Abelian gauge group is frequently referred to as a Yang–Mills theory.

The full gauge-invariant Lagrangian for Dirac spinor fields interacting with vector gauge fields is

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_{\mathrm{G}}\,, \tag{8.42}$$

where

$$\begin{aligned}\mathcal{L}_1 &= \overline{\boldsymbol{\psi}}\,(\mathrm{i}\gamma^\mu \boldsymbol{D}_\mu - m)\boldsymbol{\psi} \\ &= \overline{\boldsymbol{\psi}}\,(\mathrm{i}\gamma^\mu \partial_\mu - m)\boldsymbol{\psi} - g\,A_{i\mu}\,\overline{\boldsymbol{\psi}}\,\gamma^\mu T_i \boldsymbol{\psi}\,;\end{aligned} \tag{8.43}$$

and

$$\begin{aligned}\mathcal{L}_\mathrm{G} &= -\frac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) \\ &= -\frac{1}{4}B^i_{\mu\nu}B^{\mu\nu}_i + \frac{g}{2}\,f_{ijk}\,B^i_{\mu\nu}A^\mu_j A^\nu_k - \frac{g^2}{4}\,f_{ijk}f_{i\ell m}\,A_{j\mu}A_{k\nu}A^\mu_\ell A^\nu_m\,,\end{aligned} \tag{8.44}$$

together with the definition

$$B^i_{\mu\nu} \equiv \partial_\mu A^i_\nu - \partial_\nu A^i_\mu\,. \tag{8.45}$$

The spinor fields transform as some representation of the gauge group, but the gauge fields must belong to the adjoint representation. Since it is not possible to construct gauge-invariant mass terms, the gauge fields are necessarily massless, just as in the Abelian case. However, in contrast to the Abelian case, the part of the Lagrangian that describes the gauge fields, $\mathcal{L}_\mathrm{G}$, constitutes by itself a nontrivial interacting theory (pure Yang–Mills theory): besides the expected kinetic terms, it includes self-couplings stemming from the nonlinear expression (39), with coupling strengths that depend on the single constant $g$. The physical reason for the presence of these couplings can be easily understood: each non-Abelian gauge field $A^\mu_i$ carries a charge characteristic of the group and labeled by the index $i$, and so it must couple to every field carrying any such charge, including itself and other members of the gauge multiplet. Exactly for the same reason, gravitation is also an inherently nonlinear theory because the gravitational field interacts with everything that has energy density, including itself.

We have considered so far a *simple* Lie group as the gauge group: in this case, the generators of the group transform irreducibly under the action of the group and therefore must have the same coupling constant $g$, regardless of the representation. Basically, $g$ cannot be arbitrarily scaled, its normalization being fixed by the commutation relations characteristic of the group. This is in sharp contrast with the $\mathrm{U_Q}(1)$ case, where there are no such constraints on the coupling constant $q$, which may assume different values for different representations. If the gauge group is a direct product of simple group factors, e.g. $\mathrm{SU}(m)\times\mathrm{SU}(n)$, generators of the different factors do not mix under the action of the group, and an independent gauge coupling constant comes with each factor in the product group. Finally, let us note that we may add, in a simple generalization of the above discussion, any other matter fields belonging to any other representations of the gauge group with appropriate matrices $T_i$. In particular, it is possible to have the left- and the right-handed components of Dirac fields transforming independently as different representations of the gauge group.

## 8.4 Quantum Chromodynamics

Although historically the non-Abelian gauge principle was first used to formulate a unified theory of weak and electromagnetic interactions, the theory of strong interactions of quarks is the more obvious extension of quantum electrodynamics because the gauge symmetry on which it is based is a simple Lie group and also because the symmetry remains manifestly intact throughout. This theory marks the culmination of significant progress made over many years on two levels in the study of the physics of elementary particles.

On the one hand, major quantitative advances were achieved by the quark model in correlating detailed data in hadron spectroscopy (masses, decay rates, etc.) and by the parton model in describing the scaling phenomenon as observed in deeply inelastic, large momentum-transfer processes (such as ep $\to$ e+X and $e^+e^- \to$ hadrons). Here *parton* is the generic name given by Feynman to an independently moving constituent within a hadron, of which the quark is but an example, and *scaling* refers to the property, predicted by J. D. Bjorken, that the structure functions appearing in the cross-sections of deeply inelastic, hard processes depend only on a certain dimensionless combination of energy variables. (The structure functions give the probability of finding a parton inside a hadron carrying a certain fraction of the hadron's momentum.) The success of the quark-parton model implies that the hadron, when viewed in a frame in which its momentum is very large, is composed of almost-free constituents; in other words, quarks can interact weakly at short distances (see Chaps. 10, 12). Another key result drawn from these studies is that quarks have a three-valued quantum number, called *color*. Observations require exact color symmetry and the absence of isolated color multiplets other than singlets; this suggests that the forces between the colored quarks must be color dependent or, equivalently, they must carry 'color charges'.

On the other hand, important new ideas emerged from developments in quantum field theory. These ideas revolve around the demand of renormalizability of physical theories and the notion of energy-dependent coupling strengths. To make sense, a quantum field theory must be finite or can be made finite (renormalized) by introducing a finite number of counterterms into the original Lagrangian without changing its basic form. Renormalizability in theories involving vector quanta can be ensured by gauge invariance. In a renormalization procedure, the kinematic point at which the physical parameters, such as the mass and the coupling constant, are defined is arbitrary. However, since the physical content of the theory should remain invariant under a mere change of the normalization condition, there must be relations between physical quantities taken at different reference points. The coupling constant, for example, should be regarded as a function of the reference point and, in this sense, is energy and momentum dependent. When this effective coupling constant decreases as the relevant energy scale grows (or, equivalently, as distances shrink), the theory is said to be *asymptotically free*. Asymptotic freedom offers a possible explanation for Bjorken's scaling and is part of the reason why quarks and other hadronic constituents behave as if weakly bound inside a target nucleon, yet are not produced as free particles in final states of deep inelastic scatterings. This suggests that the field theory of strong interactions must be asymptotically free. We now know that all pure Yang–Mills theories based on groups without Abelian factors are asymptotically free, and theories of non-Abelian gauge fields and fermion multiplets are asymptotically free only if the theory does not have too many fermions. This means, for example, if the gauge group is SU(3) the number of fermion triplets is limited to sixteen or less. Another known result is that a renormalizable field theory cannot be asymptotically free unless it involves non-Abelian gauge fields. (A more detailed discussion is found in Chap. 15.)

**The QCD Lagrangian.** All this leads to the conviction that the strong interactions should be described by non-Abelian gauge fields and that it is the color symmetry that should be gauged. The resulting theory is a Yang–Mills theory based on the color SU(3) group, containing eight vector gauge bosons called *gluons*, together with different flavors of quarks, each transforming as the fundamental triplet representation. It is assumed in addition that the color gauge invariance remains exact, unbroken by any mechanism, so that the gluons remain massless. The theory, called *quantum chromodynamics* (Gross and Wilczek 1973; Fritzsch, Gell-Mann, and Leutwyler 1973; Weinberg 1973), has a Lagrangian of the form

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}\, F^i_{\mu\nu} F_i^{\mu\nu} + \sum_{A=1}^{N_{\mathrm{f}}} \bar{\boldsymbol{\psi}}^A\, (\mathrm{i}\gamma^\mu \boldsymbol{D}_\mu - m_A)\boldsymbol{\psi}^A\,, \tag{8.46}$$

where

$$\begin{aligned} F^i_{\mu\nu} &= \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_{\mathrm{s}}\, f_{ijk}\, G^j_\mu G^k_\nu\,, \\ \boldsymbol{D}_\mu \psi^A_a &= \partial_\mu \psi^A_a + \tfrac{\mathrm{i}}{2} g_{\mathrm{s}}\, G_{i\mu}\, (\lambda_i)_{ab}\, \psi^A_b\,. \end{aligned} \tag{8.47}$$

The matrices $\lambda_i$, with $i = 1, \ldots, 8$ as internal symmetry index, are the usual Gell-Mann matrices that satisfy the SU(3) Lie algebra

$$[\lambda_i,\, \lambda_j] = 2\,\mathrm{i}\, f_{ijk}\lambda_k\,. \tag{8.48}$$

The $f_{ijk}$ are the SU(3) structure constants. There are $3^2 - 1 = 8$ gluon fields, $G^i_\mu$, and $N_{\mathrm{f}} = 6$ quark color-triplets, $\psi^A_a$ with $A = 1, \ldots, N_{\mathrm{f}}$ denoting the flavors and $a = 1, 2, 3$ denoting the colors. The complete quark content of the model is

$$\psi^A_a : \quad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{8.49}$$

Since the gluons are flavor-neutral, that is, u, d, s, c, t, and b quarks have exactly the same strong interactions, the QCD Lagrangian (46) has all the flavor symmetries of a free-quark model, which are only broken by a lack of degeneracy in the quark masses. In particular, it conserves strangeness, charm, etc.. It also clearly has all the well-known strong interaction symmetries, such as invariance under charge conjugation and space inversion.

**Approaches to Solutions.** The Lagrangian (46) contains $N_\mathrm{f} + 1$ parameters: the quark masses, one for each flavor, plus one dimensionless coupling constant, $g_\mathrm{s}$. (Actually there is another parameter hidden, a vacuum angle related to the possibility of strong CP violation, which is however experimentally found consistent with zero.) With the fields second quantized, (46) forms the basis for a quantum description of the quark dynamics, and should in principle describe all the world of strong interactions. This description separates naturally into two regions: the short-distance (large invariant momentum transfer) regime in which the effective coupling strength is weak, and quarks and gluons may be treated as if they were free particles; and the large-distance (small invariant momentum transfer) regime in which the full force of the strong coupling comes into play.

In the short-distance regime, asymptotic freedom makes QCD calculable by *perturbative methods* under the right circumstances, and that is when the long-distance effects are irrelevant or can be factored out. Processes amenable to this kind of treatment include, but are not restricted to, deep inelastic lepton–hadron scattering ($\mathrm{e} + \mathrm{p} \to \mathrm{e}'$+hadrons), electron–positron annihilation ($\mathrm{e}^+\mathrm{e}^- \to$ hadrons), large invariant mass lepton-pair production ($\mathrm{p}+\mathrm{p} \to \mu^+\mu^- +$ hadrons) and jet phenomena. The successes of perturbative QCD in calculating strong interaction corrections beyond the leading order make quantitative analyses of these processes possible, and contribute to reinforcing the general belief that QCD is an essentially correct theory of the strong interaction (see Chaps. 14, 16).

The situation is much more complex in the large-distance domain. If the gluons are massless, as they are assumed to be in QCD, why have long-range strong interactions never been detected? If the strong interactions are color dependent, why are color singlets only ever observed? This is the famous outstanding problem of *color confinement*. Many methods have been devised to deal with this aspect of the strong interaction, among which the most promising consists in formulating the gauge theory on a *lattice*, in which the space-time continuum is discretized. The lattice spacing thus introduced provides a natural cutoff for momenta and allows for a natural regularization scheme in the study of the long-range properties of QCD. The gauge fields appear there as gauge-invariant dynamical variables associated with links joining adjacent lattice points; because of gauge invariance, link variables must either form closed loops or begin and end on color sources (see Fig. 8.1).

A formulation on the lattice makes feasible expansions independent of the perturbation theory; it turns out that it is in fact simpler to perform an expansion in powers of $1/g_\mathrm{s}^2$, so that the lattice gauge theory can be treated as a perturbation in the strong coupling limit. It is then possible to study various physical quantities by computer simulation based on the Monte Carlo method in which loop configurations are sampled rather than summed over. Considerable progress has been made to the point where true precision hadron mass calculations can be performed for heavy quarkonium systems and heavy–light quark systems (although the light hadron spectroscopy still eludes concerted efforts). We will not study the lattice gauge theory or any other *nonperturbative methods* of gauge theory in this book, but rather refer the reader to the series of 'Lattice' Conference Proceedings for more recent developments.

**Fig. 8.1.** (**a**) Simplest gluonic bound state; (**b**) simplest q$\bar{\text{q}}$ bound state in lattice gauge theory

**Feynman Rules for QCD.** We will end this section by giving a derivation of the Feynman rules for the tree diagrams in perturbative QCD. The rules are written in momentum space, where any field is represented by a Fourier transform of itself in space-time:

$$A(p) = \int \mathrm{d}^4x \; \mathrm{e}^{\mathrm{i}p\cdot x} A(x)\,. \tag{8.50}$$

The propagator for a quark is similar to that for the electron found in Chap. 4,

$$\mathrm{i}(S_{\mathrm{F}}(p))_{\beta\alpha}\delta_{ba}\delta_{BA} = \left(\frac{\mathrm{i}}{\not{p} - m_A + \mathrm{i}\varepsilon}\right)_{\beta\alpha} \delta_{ba}\delta_{BA}\,. \tag{8.51}$$

Each quark line is associated with three indices: $A$ for family, $a$ for color, and $\alpha$ for spinor. The quark–gluon coupling contained in (46)

$$\mathcal{L}^0_{\mathrm{h}} = -g_{\mathrm{s}}\,\bar{\psi}(x)\gamma^\mu \frac{\lambda_i}{2}\psi(x)\,G_{i\mu}(x) \tag{8.52}$$

contributes $\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}^0_{\mathrm{h}}$ to the action, which leads to the quark–gluon vertex

$$-\mathrm{i}g_{\mathrm{s}}\,(\gamma^\mu)_{\beta\alpha}\,\frac{(\lambda_i)_{ba}}{2}\,\delta_{BA}\,. \tag{8.53}$$

The pure gauge part of the Lagrangian (46) is

$$\begin{aligned}\mathcal{L}_{\mathrm{G}} &= -\frac{1}{4}\,F^i_{\mu\nu}F^{\mu\nu}_i \\ &= -\frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i + \frac{g_{\mathrm{s}}}{2}\,f_{ijk}\,G^i_{\mu\nu}G^\mu_j G^\nu_k - \frac{g^2_{\mathrm{s}}}{4}\,f_{ijk}f_{i\ell m}\,G_{j\mu}G_{k\nu}G^\mu_\ell G^\nu_m,\end{aligned} \tag{8.54}$$

where

$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu \,. \tag{8.55}$$

To derive the gluon propagator, we isolate the kinetic term in (54)

$$\mathcal{L}^0_{\mathrm{G}} = -\frac{1}{4} G_{i\mu\nu}(x)\, G^{\mu\nu}_i(x)\,, \tag{8.56}$$

which corresponds to the action

$$\begin{aligned}\int \mathrm{d}^4x\, \mathcal{L}^0_{\mathrm{G}} &= -\frac{1}{2}\int \mathrm{d}^4x\, \partial_\mu G_{i\nu}\, (\partial^\mu G^\nu_i - \partial^\nu G^\mu_i) \\ &= -\frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, G^\mu_i(-p)\, p^2 \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) G^\nu_i(p)\,. \end{aligned} \tag{8.57}$$

The integrand contains the reciprocal of the propagator. In order to invert it, one would find it convenient to introduce first the transverse and longitudinal projection operators

$$P^{\mathrm{T}}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,, \qquad P^{\mathrm{L}}_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}\,, \tag{8.58}$$

which have the properties

$$\left(P^{\mathrm{T}}_{\mu\nu}\right)^2 = P^{\mathrm{T}}_{\mu\nu}\,; \quad \left(P^{\mathrm{L}}_{\mu\nu}\right)^2 = P^{\mathrm{L}}_{\mu\nu}\,; \quad P^{\mathrm{L}}_{\mu\nu} P^{\mathrm{T}}_{\mu\nu} = 0\,; \quad P^{\mathrm{T}}_{\mu\nu} + P^{\mathrm{L}}_{\mu\nu} = g_{\mu\nu}\,. \tag{8.59}$$

Then the inverse propagator from (57) may be rewritten as

$$[D_{\mu\nu}(p)\,]^{-1} = -p^2 P^{\mathrm{T}}_{\mu\nu} + 0 P^{\mathrm{L}}_{\mu\nu}\,. \tag{8.60}$$

It is a purely transverse, singular operator, and therefore cannot be inverted. This difficulty stems from the fact that, just as for the photon, not all components of the gluon fields are physical. In order to calculate physical quantities, it is necessary to exclude the unphysical field components and select a definite gauge in which calculations are to be done. In the Lagrangian formalism, the gauge selection may be made from the start by introducing an extra term into the Lagrangian itself. This gauge-fixing term may be chosen as

$$\mathcal{L}_{\mathrm{GF}} = \frac{-1}{2\xi}\, \partial_\mu G^\mu_i\, \partial_\nu G^\nu_i = \frac{1}{2\xi} G^\mu_i \partial_\mu \partial_\nu G^\nu_i + \text{total derivative}\,, \tag{8.61}$$

where $\xi$ is a real parameter corresponding to different gauges (e.g. $\xi = 1$ for Feynman gauge and $\xi = 0$ for Landau gauge) and should not affect physical quantities. The terms quadratic in the fields in the action then yield

$$\int \mathrm{d}^4x\, (\mathcal{L}^0_{\mathrm{G}} + \mathcal{L}_{\mathrm{GF}}) = \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, G^\mu_i(-p) \left[-g_{\mu\nu}p^2 + (1-\xi^{-1}) p_\mu p_\nu\right] G^\nu_i(p)\,.$$

The inverse gluon propagator can be immediately read off:

$$D^{-1}_{\mu\nu}(p) = -g_{\mu\nu}p^2 + (1-\xi^{-1})p_\mu p_\nu \; = \; -p^2 P^{\mathrm{T}}_{\mu\nu} - \xi^{-1}p^2 P^{\mathrm{L}}_{\mu\nu} \,. \tag{8.62}$$

It is now nonsingular as long as $\xi \neq \infty$, and admits as its inverse

$$D_{\mu\nu}(p) = -\frac{1}{p^2+\mathrm{i}\varepsilon}P^{\mathrm{T}}_{\mu\nu} - \frac{\xi}{p^2+\mathrm{i}\varepsilon}P^{\mathrm{L}}_{\mu\nu} \,. \tag{8.63}$$

To each gluon internal line, we thus assign the expression

$$\mathrm{i}D_{\mu\nu}(p)\delta_{ij} = \frac{\mathrm{i}}{p^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{(1-\xi)p_\mu p_\nu}{p^2+\mathrm{i}\varepsilon}\right]\delta_{ij} \,. \tag{8.64}$$

To obtain the Feynman rules for the gluon interaction vertices, we Fourier transform the remaining terms in (54). For the three-gluon coupling we get

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}^1_{\mathrm{G}}(\text{three gluons}) =& \frac{1}{2}\, g_{\mathrm{s}} f_{ijk} \int \frac{\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4r}{(2\pi)^{12}}\,(2\pi)^4\delta^{(4)}(p+q+r) \\ &\times [g_{\lambda\nu}p_\mu - g_{\lambda\mu}p_\nu]\, G^\lambda_i(p)G^\mu_j(q)G^\nu_k(r) \,. \end{aligned} \tag{8.65}$$

Using the antisymmetry of $f_{ijk}$ in its indices and the invariance of the whole expression under simultaneous permutations of its indices, we interchange $i$, $\lambda$, $p$ with $j$, $\mu$, $q$, and $i$, $\lambda$, $p$ with $k$, $\nu$, $r$ to obtain a fully symmetrized expression

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}^1_{\mathrm{G}} =& \frac{1}{6}\, g_{\mathrm{s}} f_{ijk} \int \frac{\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4r}{(2\pi)^8}\,\delta^{(4)}(p+q+r)\, G^\lambda_i(p)G^\mu_j(q)G^\nu_k(r) \\ &\times [g_{\lambda\nu}p_\mu - g_{\lambda\mu}p_\nu - g_{\mu\nu}q_\lambda + g_{\lambda\mu}q_\nu - g_{\lambda\nu}r_\mu + g_{\mu\nu}r_\lambda] \\ =& \frac{1}{3!}\, g_{\mathrm{s}} f_{ijk} \int \frac{\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4r}{(2\pi)^8}\,\delta^{(4)}(p+q+r)\, G^\lambda_i(p)G^\mu_j(q)G^\nu_k(r) \\ &\times [g_{\lambda\nu}(p-r)_\mu + g_{\lambda\mu}(q-p)_\nu + g_{\mu\nu}(r-q)_\lambda] \,. \end{aligned} \tag{8.66}$$

The Feynman rule for three-gluon vertex can then be read off:

$$g_{\mathrm{s}}\, f_{ijk}\, [\, g_{\lambda\nu}(p-r)_\mu + g_{\lambda\mu}(q-p)_\nu + g_{\mu\nu}(r-q)_\lambda \,] \,, \tag{8.67}$$

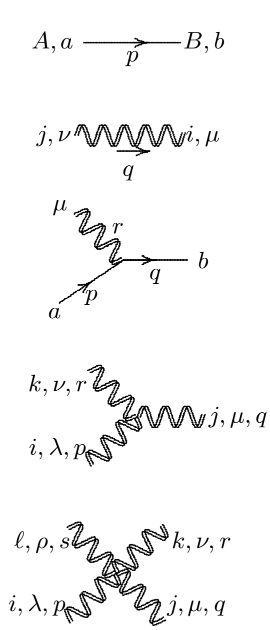
subject to four-momentum conservation

$$p + q + r = 0 \,. \tag{8.68}$$

A similar symmetrization is applied to the four-gluon coupling:

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}^2_{\mathrm{G}}(\text{four gluons}) =& \frac{1}{4}\int \frac{\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4r\,\mathrm{d}^4s}{(2\pi)^{16}}\,(2\pi)^4\delta^{(4)}(p+q+r+s) \\ &\times G^\lambda_i(p)G^\mu_j(q)G^\nu_k(r)G^\rho_\ell(s)\,(-\mathrm{i}g^2_{\mathrm{s}})\, f_{nij}f_{nk\ell}\, g_{\lambda\nu}g_{\mu\rho} \end{aligned}$$

$$\begin{aligned}=& \frac{1}{4!}\int \frac{\mathrm{d}^4p\,\mathrm{d}^4q\,\mathrm{d}^4r\,\mathrm{d}^4s}{(2\pi)^{12}}\,\delta^{(4)}(p+q+r+s)\, G^\lambda_i(p)G^\mu_j(q)G^\nu_k(r)G^\rho_\ell(s) \\ &\times (-\mathrm{i}g^2_{\mathrm{s}})\big[f_{nij}f_{nk\ell}(g_{\lambda\nu}g_{\mu\rho} - g_{\mu\nu}g_{\lambda\rho}) + f_{nkj}f_{ni\ell}(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\mu}g_{\nu\rho}) \\ &+ f_{nik}f_{nj\ell}(g_{\lambda\mu}g_{\nu\rho} - g_{\mu\nu}g_{\lambda\rho})\big] \,. \end{aligned} \tag{8.69}$$




| | |
|---|---|
| Quark propagator | $\dfrac{\mathrm{i}}{\not p - m_A + \mathrm{i}\varepsilon}\,\delta_{ba}\,\delta_{BA}$ |
| Gluon propagator | $\dfrac{\mathrm{i}\delta_{ij}}{q^2+\mathrm{i}\varepsilon}[-g_{\mu\nu} + \dfrac{(1-\xi)q_\mu q_\nu}{q^2+\mathrm{i}\varepsilon}]$ |
| Quark–gluon vertex | $-\mathrm{i}g_{\mathrm{s}}\gamma_\mu(T_i)_{ba}$ <br> $(p+r-q=0;\ T_i=\lambda_i/2)$ |
| Three-gluon vertex | $-g_{\mathrm{s}}f_{ijk}[\,g_{\lambda\mu}(p-q)_\nu + g_{\mu\nu}(q-r)_\lambda + g_{\nu\lambda}(r-p)_\mu\,]$ <br> $(p+q+r=0)$ |
| Four-gluon vertex | $-\mathrm{i}g_{\mathrm{s}}^2\,[f_{ijm}f_{k\ell m}(g_{\lambda\nu}g_{\mu\rho}-g_{\mu\nu}g_{\lambda\rho}) + f_{ikm}f_{j\ell m}(g_{\lambda\mu}g_{\nu\rho}-g_{\mu\nu}g_{\lambda\rho}) + f_{kjm}f_{i\ell m}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\nu\rho})]$ <br> $(p+q+r+s=0)$ |

**Fig. 8.2.** Feynman rules for QCD tree diagrams

This yields the Feynman rule for the four-gluon vertex:

$$(-\mathrm{i}g_{\mathrm{s}}^2)\big[f_{nij}f_{nk\ell}(g_{\lambda\nu}g_{\mu\rho}-g_{\mu\nu}g_{\lambda\rho}) + f_{nkj}f_{ni\ell}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\nu\rho}) + f_{nik}f_{nj\ell}(g_{\lambda\mu}g_{\nu\rho}-g_{\mu\nu}g_{\lambda\rho})\big] \tag{8.70}$$

with four-momentum conservation at the vertex

$$p+q+r+s=0\,. \tag{8.71}$$

The Feynman rules thus derived from the Lagrangian (46) are summarized in Fig. 8.2. However, as rules for QCD, they are not complete. In a full quantum formulation of QCD in a covariant gauge like (61), an additional, nonphysical (*ghost*) field has to be introduced, whose main effect is to suppress the nontransverse components of real gluons while preserving gauge invariance (see Chap. 15). A complete list of the Feynman rules for QCD is given in the Appendix.

## 8.5 Spontaneous Breaking of Global Symmetries

Experiment shows that quantum electrodynamics is a gauge theory consistent in all aspects with the principle of gauge invariance applied to the $U_{\mathrm{Q}}(1)$ group. In particular, the photon can be identified with the massless gauge field of the group and interacts just as expected with the conserved fermion current that follows from the gauge symmetry.

On the other hand, in spite of its apparently distinctive properties (e.g. a much shorter force range, a greater diversity in transition modes), the weak interaction gives clear hints to its close parentage with the electromagnetic interaction. In particular, the currents found in many weak processes are electrically charged and have precisely the form implied by a non-Abelian symmetry based on a certain semisimple group. It is thus quite possible that there exists a gauge theory that can describe both weak and electromagnetic interactions. However, as we have seen earlier in this chapter, the gauge fields required by gauge invariance must apparently be massless and must therefore generate long-range forces. In order to construct a gauge theory of this kind for weak interactions, one is then confronted with the problem of reconciling the presence of massive gauge fields needed to generate the short-range weak forces actually observed with the preservation in some sense of gauge invariance essential for a renormalizable theory.

One way of generating masses for vector bosons without destroying the underlying gauge symmetry of the theory is by 'spontaneously' breaking that symmetry. This phrase refers to a process in which, from a set of degenerate minimum energy states that are equivalent by symmetry, one arbitrarily selects a member of the multiplet as *the* physical ground state of the system in apparent violation of the underlying symmetry. But in reality the symmetry is not lost in the process, it is merely hidden and can be recovered through special relations between masses and couplings. That it is possible to pick the ground state in this way simply reflects the fact, fairly widespread in nature, that physical states may exist with a symmetry apparently lower than that of the basic equations of motion.

### 8.5.1 The Basic Idea

To understand the idea of spontaneous breakdown of symmetry, let us mentally consider a large sample of ferromagnetic material at 0° K in the absence of any external field. A ferromagnet is viewed in the Heisenberg model as an infinite regular array of spin-$\frac{1}{2}$ magnetic dipoles with spin–spin interactions between nearest neighbors such that neighboring dipoles tend to align. Although the Hamiltonian describing the system is rotationally invariant, the ground state is not always. At high temperatures, thermal agitations will make the magnetic moments flutter at random in different directions, so that there is no net magnetization, which results in a rotationally symmetric state endowed with the same symmetry as the law of interaction. If the ferromagnet is now sufficiently cooled down (below a certain critical temperature, called the Curie temperature), all the atomic dipole moments will tend to align parallel to each other and to some arbitrary direction, leading to a nonzero magnetization for the sample. This is one of the infinitely many degenerate lowest-energy states that exist for an infinite ferromagnet, and the symmetry resides hidden in the equivalence of these states through rotations. Transitions between these states are not possible, because for an infinite ferromagnet any single transition would require an infinite amount of energy. The particular ground state the system 'spontaneously' falls into as it cools down cannot be foreseen, and certainly is not symmetric since the magnetization points in a definite direction; it corresponds to a magnetization vector $\boldsymbol{M}$ with magnitude $M$ such that the free energy $F$ of the ferromagnet is minimum, as shown in Fig. 8.3.

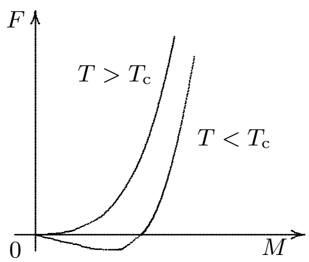



**Fig. 8.3.** Free energy $F$ of a ferromagnet as function of magnetization $M$

We now attempt to transfer this insight to relativistic quantum mechanics, substituting a particle Hamiltonian (or Lagrangian) for the ferromagnet Hamiltonian, the particle vacuum for the ferromagnet ground state and some other symmetry for rotational symmetry. Specifically, assuming nature to possess symmetries that are not manifest to us because we live in an imperfectly symmetric universe, we will take the Lagrangian that defines the particle theory to be invariant in some *internal symmetry*, but the particle vacuum to be lacking this symmetry. This asymmetric state is realized by requiring that the vacuum-to-vacuum expectation value of some field be nonvanishing, much as the ferromagnetic ground state was determined by a nonzero magnetization. The field in question cannot have a nonzero spin because otherwise the vacuum would be characterized by a nonzero angular momentum and rotational invariance would have been broken. Since the vacuum is observed to be rotationally invariant the field must be spinless, and the internal symmetry of the theory must be broken by a scalar field acquiring a nonzero vacuum expectation value. As translation invariance is also an observed symmetry of particle physics, this expectation value must not depend on space-time in the absence of any source. The basic conjecture is that there exist, beside matter and gauge fields, one or more spin-0 fields, called the *Higgs fields*, which would assume uniform nonzero values even in the vacuum and which could couple to each other and to other, massless particles to give them masses.

A quantum-mechanical vacuum is a complex state filled with pairs of virtual particles and antiparticles continuously being created and annihilated. If those virtual particles interact strongly enough among themselves, they might form a permanent state of high density, called a *vacuum condensate*.

There is a thermodynamic transition point separating the vacuum without a condensate from the vacuum with a condensate. For a condensate to form, there must be a strong enough attraction among the particles at low density, but also a strong enough repulsion at high density to prevent a runaway situation to occur. It is believed that such a situation exists in the world of the fundamental particles, where the vacuum is filled with a high density of Higgs fields, the *Higgs condensate*. By interacting with the light particles (bosons and fermions) that populate this vacuum, the condensate drags them sufficiently down to make them massive. Such is in simplified terms the physics of the *Higgs mechanism*, as this mass-generating process is called.

### 8.5.2 Breakdown of Discrete Symmetry

Let us begin with the simplest model having a discrete symmetry and containing a single massless *real* scalar field which plays the role of the Higgs field. The part of the Lagrangian relevant to the present discussion is

$$\mathcal{L}_{\mathrm{s}} = \tfrac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi - V(\phi)\,, \tag{8.72}$$

with a potential parameterized by two real constants, $\lambda$ and $\mu^2$,

$$V(\phi) = \frac{1}{2}\,\mu^2\,\phi^2 + \frac{\lambda}{4!}\,\phi^4\,. \tag{8.73}$$

The corresponding energy density is given by

$$\mathcal{H}_{\mathrm{s}} = \tfrac{1}{2}\, \left[(\partial_0\phi)^2 + (\nabla\phi)^2\right] + V(\phi)\,. \tag{8.74}$$

The only internal symmetry of the model is its invariance to field reflection

$$\phi(x) \to \phi'(x) \equiv -\phi(x)\,. \tag{8.75}$$

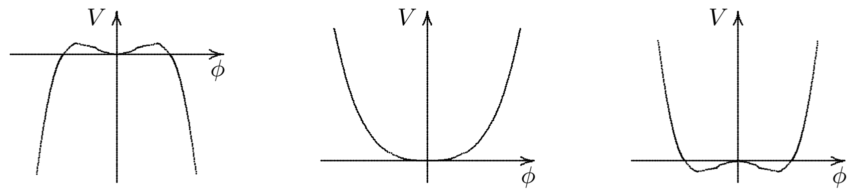
The energy minimum of the system is determined at the classical level by the condition

$$\frac{\partial V}{\partial \phi} = \phi\,\left(\mu^2 + \frac{\lambda}{6}\phi^2\right) = 0\,. \tag{8.76}$$

When $\lambda < 0$ the potential $V(\phi)$ has no stable minima for finite $\phi$ (see Fig. 8.4a). We will therefore assume $\lambda \geq 0$. Then, for $\mu^2 > 0$ the potential has a unique minimum at $\phi = 0$ (as shown in Fig. 8.4b) and the symmetry of the vacuum is manifest. More interesting is the case $\mu^2 < 0$ when $V$ has minima at the nonzero field values

$$\phi = \pm\sqrt{-6\mu^2/\lambda} \tag{8.77}$$

(shown in Fig. 8.4c). This is precisely the situation in which $V$ is attractive at small values of $\phi$ but becomes strongly repulsive at large values. The field




(**a**) $\lambda < 0$ (**b**) $\lambda > 0,\ \mu^2 > 0$ (**c**) $\lambda > 0,\ \mu^2 < 0$

**Fig. 8.4a–c.** Potential for the scalar field with reflection symmetry, for different values of the parameters

values given in (77) are independent of $x$ and correspond to the quantum-mechanical vacuum expectation value of the field operator, denoted by $\langle\phi\rangle$ or $\langle 0\,|\,\phi\,|\,0\rangle$. Because of the reflection symmetry (75) of the model, whichever solution is chosen will lead to the same physics; but once the choice is made, the symmetry of the system is (spontaneously) broken. Let us arbitrarily select the positive value of $\phi$ at minimum $V$ to define the vacuum:

$$\langle\phi\rangle = v = \sqrt{-6\mu^2/\lambda}\,. \tag{8.78}$$

In order to do any calculations beyond the ground state, it is convenient to introduce a new field

$$\chi(x) = \phi(x) - v\,, \tag{8.79}$$

which is designed to have a zero vacuum expectation value. It measures field oscillations about the uniform background $\phi = v$. In terms of $\chi$, the Lagrangian density becomes

$$\mathcal{L}_\mathrm{s} = \frac{1}{2}\left[\partial_\mu\chi\,\partial^\mu\chi - (-2\mu^2)\chi^2\right] - \frac{\lambda}{4!}\left(4v\,\chi^3 + \chi^4\right) - \frac{1}{4}\,\mu^2 v^2\,. \tag{8.80}$$

It can now be interpreted in the usual way, with no more concern about the properties of the vacuum, since the dynamic field vanishes in the vacuum, $\langle\chi\rangle = 0$. It simply describes the dynamics of a spin-0 field with real mass $\sqrt{-2\mu^2}$. Even though it is the same Lagrangian as before, the presence of the cubic term $\chi^3$ gives us no reason to suspect that a symmetry actually lies in the background.

### 8.5.3 Breakdown of Abelian Symmetry

We are eventually interested in theories with continuous gauge symmetries. The simplest model that exhibits such a symmetry is the *complex* scalar field theory described by the Lagrangian

$$\begin{aligned} \mathcal{L}_\mathrm{s} &= \partial_\mu\varphi\,\partial^\mu\varphi^* - V(\varphi,\varphi^*)\,, \\ V(\varphi,\varphi^*) &= \mu^2\,\varphi\varphi^* + \tfrac{1}{4}\,\lambda\,(\varphi\varphi^*)^2\,, \end{aligned} \tag{8.81}$$

which is evidently invariant under a *global* phase transformation with an arbitrary real constant $\alpha$.

$$\begin{aligned} \varphi &\mapsto e^{-i\alpha}\varphi\,, \\ \varphi^* &\mapsto e^{i\alpha}\varphi^*\,. \end{aligned} \tag{8.82}$$

For the same reason as before, we assume $\lambda > 0$. Then, for $\mu^2$ positive, $V$ acquires an absolute minimum at $\varphi = 0$ and the vacuum has manifestly the same symmetry as the Lagrangian. But if $\mu^2$ is negative, the system has the lowest energy for

$$|\varphi|^2 = -2\mu^2/\lambda\,.$$

Thus, there is an infinite number of degenerate minima lying on a circle of radius $\sqrt{-2\mu^2/\lambda}$ (see Fig. 8.5) and differing from one another by a relative phase factor, but all equivalent through the phase transformations (82) and all leading to the same physics. Any particular choice of $\langle\varphi\rangle$ will spontaneously break the symmetry; so we may as well let its phase-angle be zero and select the vacuum such that

$$\langle\varphi\rangle = \frac{v}{\sqrt{2}} \tag{8.83}$$

for $v = \sqrt{-4\mu^2/\lambda}$. It is significant to note that in this choice only the real part of $\varphi$ acquires a nonzero vacuum expectation value, fixing the direction of the symmetry breakdown.

We now define a shifted complex field $\chi$ such that

$$\varphi = \langle\varphi\rangle + \tfrac{1}{\sqrt{2}}\,\chi \;= \tfrac{1}{\sqrt{2}}\,(v + \chi_1 + i\chi_2)\,. \tag{8.84}$$

Both real fields $\chi_1$ and $\chi_2$ have zero vacuum expectation values. They measure excitations of the fields from the vacuum in the directions radial and tangential to the circle of degenerate minima. In terms of these fields, we have for the potential

$$V = -\mu^2\chi_1^2 + \frac{\lambda}{16}(\chi_1^2 + \chi_2^2)\left[4v\chi_1 + \chi_1^2 + \chi_2^2\right] + \frac{\mu^2 v^2}{4}\,, \tag{8.85}$$

and for the Lagrangian

$$\begin{aligned} \mathcal{L}_s = &\frac{1}{2}\left[\partial_\mu\chi_1\,\partial^\mu\chi_1 - (-2\mu^2)\chi_1^2\right] + \frac{1}{2}\,\partial_\mu\chi_2\,\partial^\mu\chi_2 \\ &- \frac{\lambda}{16}\,(\chi_1^2 + \chi_2^2)\,(4v\chi_1 + \chi_1^2 + \chi_2^2) - \frac{1}{4}\,\mu^2 v^2\,. \end{aligned} \tag{8.86}$$

Evidently, the phase symmetry has been spontaneously broken. The field $\chi_1$, which represents fluctuations in the direction of symmetry breakdown,

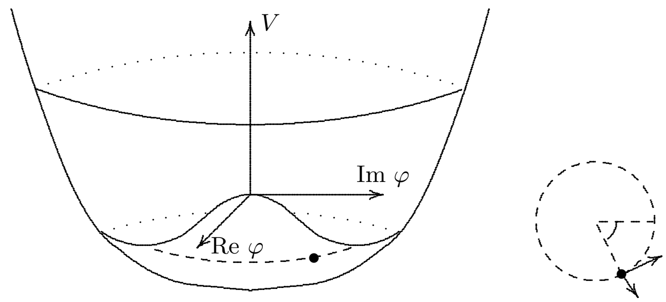



**Fig. 8.5.** Symmetry-breaking ground state in a potential that exhibits invariance under continuous symmetry transformations

acquires a mass, just as in the real scalar model. But the field $\chi_2$, which measures deviations in the direction of symmetry conservation, remains massless – a new feature, absent when it is a discrete symmetry that breaks down. In geometrical terms, as the vacuum is selected at some point on the circle of degenerate minima of $V$, it is an absolute minimum for the potential curve in the radial direction, and excitations from such a point always require energy, which implies massive modes. On the other hand, the selected vacuum has precisely the same potential energy $(-\mu^4/\lambda)$ as any other minimum, and deviations in the tangential direction, in which $V$ is flat and the total energy constant, describe the zero-frequency motion around the minimum circle. Such massless and spinless modes that arise from a spontaneous breaking of a continuous symmetry are called the *Nambu–Goldstone bosons* in particle physics. This property is not particular to the present model but is a general feature of spontaneous breakdown of gauge symmetry.

### 8.5.4 Breakdown of Non-Abelian Symmetry

Let us turn now to a general non-Abelian gauge symmetry G. Since a complex representation can always be replaced by a real one by doubling the basis vectors of the space on which it is defined, we need to consider only real representations. Thus, we take $n$ *real* scalar fields, $\phi_1, \ldots, \phi_n$, to form a column vector $\boldsymbol{\phi}$ which transforms as a (generally reducible) representation of G:

$$\boldsymbol{\phi} \to \boldsymbol{\phi}' = U\,\boldsymbol{\phi}\,, \tag{8.87}$$

where $U$ is a real, orthogonal $n \times n$ constant matrix. In the usual parameterization we write it as

$$U = \mathrm{e}^{-\mathrm{i}g\omega_j T_j}\,, \tag{8.88}$$

where $g$ and $\omega_j$ are real constants and $T_j$ for $j = 1, \ldots, N$ are the $n \times n$ matrices satisfying the Lie algebra associated with the group. These matrices are Hermitian, $T_j^\dagger = T_j$, because $U$ is unitary; as well as imaginary and antisymmetric, $T_j^* = T_j^{\mathrm{T}} = -T_j$, because $U$ is also real (an upper index T denotes a transposed matrix).

The Lagrangian for these fields is taken to be

$$\begin{aligned} \mathcal{L}_{\mathrm{s}} &= \tfrac{1}{2}\,\partial_\mu \boldsymbol{\phi}\,\partial^\mu \boldsymbol{\phi} - V(\boldsymbol{\phi})\,, \\ V(\boldsymbol{\phi}) &= \tfrac{1}{2}\mu^2\,\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi} + \tfrac{1}{16}\,\lambda\,\left(\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\right)^2\,, \end{aligned} \tag{8.89}$$

where $\lambda > 0$. As in previous cases, nothing noteworthy happens when $\mu^2$ is positive; but when $\mu^2$ turns negative, $V$ acquires an infinite set of degenerate nonzero minima at

$$|\boldsymbol{\phi}|^2 = -\frac{4\mu^2}{\lambda}\,. \tag{8.90}$$

The group symmetry is spontaneously broken when the vacuum is selected, such that, for example,

$$\langle\boldsymbol{\phi}\rangle = \boldsymbol{v}\,, \tag{8.91}$$

for some real constant $n$-dimensional vector satisfying $|\boldsymbol{v}|^2 = -4\mu^2/\lambda$. Proceeding as before, we define the shifted field

$$\boldsymbol{\chi} = \boldsymbol{\phi} - \boldsymbol{v}\,, \tag{8.92}$$

which has a zero vacuum value, $\langle\boldsymbol{\chi}\rangle = 0$. In terms of $\boldsymbol{\chi}$ the potential becomes

$$V = \frac{1}{4}\,\mu^2\boldsymbol{v}^2 + \frac{\lambda}{4}\,\left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{v}\right)^2 + \frac{\lambda}{16}\,\left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\right)\left[\,4\,\left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{v}\right) + \left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\right)\,\right]\,, \tag{8.93}$$

and the Lagrangian assumes the form

$$\begin{aligned} \mathcal{L}_{\mathrm{s}} =& \frac{1}{2}\,\left[\partial_\mu\chi_a\partial^\mu\chi_a - \tfrac{1}{2}\,\lambda\,v_a v_b\,\chi_a\chi_b\right] \\ &- \frac{\lambda}{16}\,\left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\right)\left[4\,\left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{v}\right) + \left(\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\right)\,\right] - \frac{1}{4}\,\mu^2\boldsymbol{v}^2\,. \end{aligned} \tag{8.94}$$

The masses of the fields are not apparent from (94) because they reside in the nondiagonalized quadratic terms which give the squared-mass matrix

$$\left(\mathcal{M}_{\mathrm{B}}^2\right)_{ab} = \tfrac{1}{2}\,\lambda\,v_a v_b\,, \qquad \text{for} \quad a,\, b = 1, \ldots, n\,. \tag{8.95}$$

To find the allowed eigenvalues of $\mathcal{M}_{\mathrm{B}}^2$, let it operate on any vector $T_i\,\boldsymbol{v}$:

$$\begin{aligned} \mathcal{M}_{\mathrm{B}}^2\,T_i\,\boldsymbol{v} &= \tfrac{1}{2}\,\lambda\,\boldsymbol{v}\,(\boldsymbol{v}^{\mathrm{T}}T_i\boldsymbol{v}) \;= \tfrac{1}{2}\,\lambda\,\boldsymbol{v}\,(\boldsymbol{v}^{\mathrm{T}}T_i\boldsymbol{v})^{\mathrm{T}} \\ &= \tfrac{1}{2}\,\lambda\,\boldsymbol{v}\,(\boldsymbol{v}^{\mathrm{T}}T_i^{\mathrm{T}}\boldsymbol{v}) \;= -\tfrac{1}{2}\,\lambda\,\boldsymbol{v}\,(\boldsymbol{v}^{\mathrm{T}}T_i\boldsymbol{v})\,, \end{aligned} \tag{8.96}$$

so that

$$\mathcal{M}_{\rm B}^2 \, T_i \, \boldsymbol{v} = 0 \,, \qquad \text{for} \quad i = 1, \ldots, N \,. \tag{8.97}$$

On the other hand, since the symmetry is broken by setting $\langle \boldsymbol{\phi} \rangle = \boldsymbol{v}$,

$$\boldsymbol{v} \neq U \boldsymbol{v} \approx \boldsymbol{v} - \mathrm{i} g \omega_j T_j \, \boldsymbol{v} \,,$$

and so there must exist at least one $T_k$ such that

$$T_k \, \boldsymbol{v} \neq 0 \,. \tag{8.98}$$

For each such $T_k$, the matrix $\mathcal{M}_{\rm B}^2$ has a zero-eigenvalue, as required by (97); this zero-eigenvalue corresponds to a Nambu–Goldstone mode.

Let S be the maximum subgroup of G that survives as a symmetry of the vacuum after the breakdown of G; let $M$ $(M \leq N)$ be its dimension. We can always choose the generators $T_i$ of G such that the first $M$ generators, $T_j$ for $j = 1, \ldots, M$, generate S. Then, since the vacuum remains invariant under subgroup S,

$$T_j \, \boldsymbol{v} = 0 \,, \qquad \text{for} \quad j = 1, \ldots, M \,; \tag{8.99}$$

but for the remaining generators,

$$T_k \, \boldsymbol{v} \neq 0 \,, \qquad \text{for} \quad k = M+1, \ldots, N \,, \tag{8.100}$$

and (97) tells us that $\mathcal{M}_{\rm B}^2$ admits $N - M$ zero-eigenvalues. Since the $N - M$ vectors $T_k \boldsymbol{v}$, for $k = M+1, \ldots, N$, are evidently linearly independent, there must be $N - M$ massless Nambu–Goldstone bosons in the theory, one for each symmetry-breaking generator. The other $(n - N + M)$ bosons in the system have, in general, nonvanishing masses.

**Example 8.1 Orthogonal Group**

The orthogonal group $\mathrm{G} = \mathrm{O}(n)$ has $N = \frac{1}{2} n(n-1)$ generators. We take $n$ real scalar fields to form the $n$-dimensional vector representation $\boldsymbol{\phi}$, and let their potential $V$ acquire a minimum for $|\boldsymbol{\phi}|^2 = v^2$. Among the infinite number of possible minima, a particular vector $\boldsymbol{v}$ of squared modulus $v^2$ is chosen to define the vacuum. The vacuum symmetry consists of all rotations that leave $\boldsymbol{v}$ invariant. These are the rotations that act on a space with one less dimension, and together form an orthogonal group $\mathrm{O}(n-1)$ with $M = \frac{1}{2}(n-1)(n-2)$ independent generators. In particular, if we choose the axes in the representation space such that the vacuum vector $\boldsymbol{v}$ points along the $n$th axis, so that $v_a = v \delta_{an}$, the elements of $\mathrm{O}(n-1)$ do not mix the $n$th component of $\boldsymbol{v}$ with the others. If $L_{ij}$ denote the generators of $\mathrm{O}(n)$,

$$(L_{ij})_{ab} = -\mathrm{i}(\delta_{ia} \, \delta_{jb} - \delta_{ib} \, \delta_{ja}) \,, \qquad \text{for} \quad i, j, \, a, b = 1, \ldots, n \,,$$

the vacuum vector $\boldsymbol{v}$ satisfies the conditions

$$\begin{aligned} (L_{ij}\boldsymbol{v})_a &= \ 0 && \text{for} \quad i,j = 1,\ldots,n-1\,; \\ (L_{kn}\boldsymbol{v})_a &= -\mathrm{i}v\,\delta_{ka} && \text{for} \quad k = 1,\ldots,n-1\,. \end{aligned}$$

It follows that $L_{ij}$ with $i,j = 1,\ldots,n-1$ generate the vacuum symmetry group, while $L_{kn}$ for $k = 1,\ldots,n-1$ lead to nontrivial vectors when applied on $\boldsymbol{v}$. There are, as expected, $N - M = n-1$ massless Nambu–Goldstone bosons; and since we started out with $n$ fields in all, there remains just one Higgs boson with mass $M_{\mathrm{H}}^2 = \lambda\,v^2/2$, given by the single element of $\mathcal{M}_{\mathrm{B}}^2$.

Up to now we have parameterized field deviations from the vacuum in the obvious way, that is, as in (92). Another possibility which might come handy can be illustrated by the present example. Let us start with $\boldsymbol{\phi} = \boldsymbol{v} + \boldsymbol{\chi}$ as in (92), with $v_a = v\,\delta_{an}$ for $a = 1,\ldots,n$, and construct the $n\times n$ matrix

$$U(\omega) = \exp\left(-\mathrm{i}\sum_{k=1}^{n-1}\omega_k\,L_{kn}\right).$$

Under this rotation, $\boldsymbol{\phi}$ transforms into

$$\boldsymbol{\phi}' = U\,\boldsymbol{\phi} = U\,(\boldsymbol{v} + \boldsymbol{\chi})\,.$$

Assuming that both the fluctuations $\chi_a$ and the transformation parameters $\omega_i$ are infinitesimal, we obtain up to linear terms

$$\begin{aligned} \phi'_a &\approx v_a + \chi_a - \mathrm{i}\sum_k \omega_k (L_{kn})_{ab}\,v_b \\ &\approx (v+\chi_n)\,\delta_{an} + (\chi_a - v\,\omega_a)(1-\delta_{an})\,, \qquad a = 1,\ldots,n. \end{aligned}$$

Thus, if we choose $\omega_a = \chi_a/v$, the transformed field $\boldsymbol{\phi}'$ will align with the $n$th axis, in the same direction as $\boldsymbol{v}$, so that

$$\phi'_a \approx (v+\chi_n)\,\delta_{an}\,.$$

Inversely, a general vector $\boldsymbol{\phi}$ may be obtained from the vector with components $(v+\chi_n)\,\delta_{an}$ by the rotation $U(-\omega)$. To summarize, an alternative to (92) is the parameterization

$$\boldsymbol{\phi} = \exp\left(\frac{\mathrm{i}}{v}\sum_{k=1}^{n-1}\xi_k L_{kn}\right)\boldsymbol{\phi}_{\parallel}\,, \tag{8.101}$$

where $\boldsymbol{\phi}_{\parallel}$ is an $n$-component vector with a single nonvanishing component, $(\boldsymbol{\phi}_{\parallel})_a = (v+\eta)\,\delta_{an}$. The two parameterizations are equivalent to first order, $\eta \approx \chi_n$ and $\xi_k \approx \chi_k$ for $k = 1,\ldots,n-1$. ■

## 8.6 Spontaneous Breaking of Local Symmetries

The Nambu–Goldstone bosons have the amazing property that, when it is a *local* gauge symmetry that is spontaneously broken, they disappear and simultaneously the normally massless gauge fields become massive, giving the associated long-range gauge forces a finite range. This shielding effect is akin to the Meissner effect in superconductivity, which makes an external magnetic field attenuate beyond a surface layer inside a superconductor.

### 8.6.1 Abelian Symmetry

We first study the simple Abelian model of scalar electrodynamics; when spontaneously broken, it is called the *Higgs model*. Even though it does not provide practically useful results, it will illustrate many of the ideas to be found in a more general model. The model is defined by

$$\mathcal{L} = D_\mu\varphi\, D^\mu\varphi^* - \mu^2\varphi\varphi^* - \frac{1}{4}\lambda\,(\varphi\varphi^*)^2 - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}\,, \tag{8.102}$$

where $\varphi$ is a complex scalar field, with covariant derivatives

$$\begin{aligned} D_\mu\varphi &= (\partial_\mu + \mathrm{i}qA_\mu)\varphi\,, \\ D_\mu\varphi^* &= (\partial_\mu - \mathrm{i}qA_\mu)\varphi^*\,, \end{aligned} \tag{8.103}$$

and $F_{\mu\nu}$ is the gauge-invariant field strength associated with the gauge field $A_\mu$. This Lagrangian is, of course, the version of (81) made invariant under the $U(1)$ local gauge transformations

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\omega\,, \tag{8.104}$$

$$\varphi \to \varphi' = \mathrm{e}^{-\mathrm{i}q\omega}\,\varphi\,. \tag{8.105}$$

When $\mu^2$ is positive the Lagrangian (102) just describes a scalar particle of mass $\mu$ and charge $q$ interacting with an electromagnetic field. We are rather interested in the case of negative $\mu^2$ when the potential develops minima at the field values $|\varphi|^2 = -2\mu^2/\lambda$. Then the symmetry may be hidden by selecting the vacuum so that the field acquires the vacuum expectation value

$$\langle\varphi\rangle = \tfrac{1}{\sqrt{2}}\,v\,, \tag{8.106}$$

for the real number $v = \sqrt{-4\mu^2/\lambda}$. Now, define the real fields $\chi_1$ and $\chi_2$ through

$$\varphi(x) = \tfrac{1}{\sqrt{2}}(v + \chi_1 + \mathrm{i}\chi_2)\,. \tag{8.107}$$

Then the covariant derivative of the field becomes

$$D_\mu\varphi = \frac{1}{\sqrt{2}}\left[\partial_\mu\chi_1 + \mathrm{i}qv\,(A_\mu + \frac{1}{qv}\partial_\mu\chi_2) + \mathrm{i}qA_\mu\,(\chi_1 + \mathrm{i}\chi_2)\right]\,, \tag{8.108}$$

leading to the expression for the kinetic term

$$\begin{aligned} K &= D_\mu\varphi\, D^\mu\varphi^* \\ &= \frac{1}{2}\left(\partial_\mu\chi_1 - qA_\mu\chi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\chi_2 + qvA_\mu + qA_\mu\chi_1\right)^2 \ . \end{aligned} \tag{8.109}$$

As in the complex scalar model with global symmetry, here $\chi_1$ acquires a mass too, but a clear interpretation of $\chi_2$ and $A_\mu$ is difficult to have because they are coupled together in the second order. What is significant is that this coupling comes as part of the expression

$$\frac{1}{2}\,(qv)^2\left(A_\mu + \frac{1}{qv}\partial_\mu\chi_2\right)\left(A^\mu + \frac{1}{qv}\partial^\mu\chi_2\right) , \tag{8.110}$$

which could be regarded as a mass term for a redefined vector field

$$A'_\mu = A_\mu + \frac{1}{qv}\partial_\mu\chi_2\,. \tag{8.111}$$

This field redefinition appears as a gauge transformation (104) of $A_\mu$ with the local transformation parameter $\omega = \chi_2/qv$; it tells us that $\chi_2$ has no real physical significance and might be eliminated by an appropriate gauge transformation. With this in mind, let us rewrite the gauge transformation (105) for the real fields $\chi_i$ :

$$\begin{aligned} \chi_1 \to \chi'_1 &= -v + (\cos q\omega)(v+\chi_1) + (\sin q\omega)\chi_2\,, \\ \chi_2 \to \chi'_2 &= (\cos q\omega)\chi_2 - (\sin q\omega)(v+\chi_1)\,. \end{aligned} \tag{8.112}$$

For an infinitesimal $\omega$ this gives

$$\begin{aligned} \chi'_1 &\approx \chi_1 + q\omega\,\chi_2\,, \\ \chi'_2 &\approx \chi_2 - q\omega\,\chi_1 - q\omega\, v\,. \end{aligned} \tag{8.113}$$

We see that the field $\chi_2$ transforms with an inhomogeneous term, just like $A_\mu$, so that separately neither can have a direct physical meaning. In fact the gauge invariance of the theory allows us to make a gauge transformation that completely removes $\chi_2$. It suffices to choose as parameter

$$\omega(x) = \frac{1}{q}\tan^{-1}\left[\frac{\chi_2}{v+\chi_1}\right] . \tag{8.114}$$

In this gauge, only two fields survive: $\chi'_1$, which will be renamed $H$, and

$$A'_\mu = A_\mu + \partial_\mu\omega \approx A_\mu + \frac{1}{qv}\partial_\mu\chi_2 + \text{higher-order terms}\,, \tag{8.115}$$

which will be simply called $A_\mu$. We then have, for the potential,

$$V = -\mu^2 H^2 + \frac{1}{16}\lambda H^2 (4vH + H^2) + \frac{1}{4}\mu^2 v^2 , \tag{8.116}$$

and, for the Lagrangian,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}[\partial_\mu H \,\partial^\mu H + 2\mu^2 H^2] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(qv)^2 A_\mu A^\mu \\ &+ \frac{1}{2}q^2 A_\mu A^\mu H(H+2v) - \frac{1}{16}\lambda H^3(H+4v) - \frac{1}{4}\mu^2 v^2 .\end{aligned} \tag{8.117}$$

This result can now be naturally interpreted as the Lagrangian for a neutral scalar particle of mass $\sqrt{-2\mu^2}$ and a massive vector particle with mass $M_\mathrm{A} = qv$, conveniently decoupled from each other in the second order. The would-be Goldstone boson is completely gone; it has been gauged away, absorbed as the newly formed longitudinal polarization state of the vector field, as indicated by (115). Thus, two massless particles have been disposed of: the vector meson has gained mass and the Goldstone boson has been eliminated. Instead of a massless gauge boson with its two transverse modes and a complex scalar field composed of two real components, we have, after the symmetry breaking, a single real spin-0 field $H$ and a massive spin-1 meson with three spin states (two transverse and one longitudinal). The number of degrees of freedom has not changed; it remains four.

In the gauge specified by (114) all fields that survive the symmetry breakdown are physical fields; fictitious particles, whose Green's functions would have singularities that violate unitarity, are absent. But the Lagrangian (117) contains a massive vector field, whose propagator for large momentum grows as $1/M_\mathrm{A}^2$ rather than as $k^2$ characteristic of massless vector fields and, therefore, does not lead to an obviously renormalizable theory. This gauge, manifestly unitary but not manifestly renormalizable, is called the *unitary* (or U) gauge. A surprising result, obtained by G. 't Hooft, is that the renormalizability of the theory, though not manifest in (117), has in fact been preserved in the spontaneous symmetry breaking; it is not apparent simply because of the particular gauge being used. 't Hooft's proof of renormalizability of spontaneously broken gauge theories relies on the discovery that it is useful to adopt a class of more general gauges, called $\mathrm{R}_\xi$, which even though not manifestly unitary, are explicitly renormalizable; that is, the ultraviolet divergences that arise will behave no worse than those occurring in QED.

The $\mathrm{R}_\xi$-gauges may be enforced by adding to the Lagrangian (117) the gauge-fixing term

$$\begin{aligned}\mathcal{L}_\mathrm{GF} &= -\frac{1}{2\xi}(\partial_\mu A^\mu - f)^2 \\ &= -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{\xi}(\partial_\mu A^\mu)\, f - \frac{1}{2\xi} f^2 ,\end{aligned} \tag{8.118}$$

where $\xi$ is a positive real constant that defines the gauge, while $f$ will be chosen so as to cancel the awkward quadratic coupling of $A_\mu$ and $\chi_2$ found in (109). This is done by requiring

$$\xi^{-1}(\partial_\mu A^\mu)\, f + qv\, A^\mu \partial_\mu \chi_2 = 0\,,$$

which is satisfied up to a total derivative, provided that

$$f = \xi\, qv\, \chi_2\,. \tag{8.119}$$

Together with the gauge-fixing term, the Lagrangian becomes in this gauge

$$\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}\big[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2 + (qv)^2 A_\mu^2 + 2qv\, A^\mu\partial_\mu\chi_2 \\
& + 2q\, A^\mu(\chi_1\partial_\mu\chi_2 - \chi_2\partial_\mu\chi_1) + q^2\, A_\mu^2(2v\chi_1 + \chi_1^2 + \chi_2^2)\big] \\
& - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 - \frac{1}{2\xi}(\xi qv)^2\, \chi_2^2 \\
& + \mu^2\chi^2 - \frac{\lambda}{16}(\chi_1^2 + \chi_2^2)(4v\chi_1 + \chi_1^2 + \chi_2^2) - \frac{\mu^2 v^2}{4} \\
= {} & \frac{1}{2}\,\big[(\partial_\mu\chi_1)^2 + 2\mu^2\chi_1^2\big] + \frac{1}{2}\,\big[(\partial_\mu\chi_2)^2 - \xi(qv)^2\chi_1^2\big] \\
& - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 + \frac{1}{2}(qv)^2 A_\mu^2 + \text{higher-order terms.}
\end{aligned} \tag{8.120}$$

Thus, in a general $\mathrm{R}_\xi$-gauge three fields are involved, decoupled from each other in the second order: the vector field of mass $M_A = qv$, the Higgs boson of mass $M_H = \sqrt{-2\mu^2}$, and the former Goldstone boson now with mass $\sqrt{\xi}\, M_A$. The dependence of the latter on the gauge parameter reveals the inherently nonphysical character of the Goldstone field.

The propagators of these fields can now be found from their respective quadratic terms in (120). Thus, the Goldstone mode has the propagator

$$\Delta(p, \sqrt{\xi} M_\mathrm{A}) = [\, p^2 - \xi M_\mathrm{A}^2]^{-1}\,, \tag{8.121}$$

and the Higgs boson has the propagator

$$\Delta(p, M_\mathrm{H}) = [\, p^2 - M_\mathrm{H}^2 + \mathrm{i}\varepsilon]^{-1}\,. \tag{8.122}$$

To find the propagator for the vector field, we consider the quadratic terms in $A_\mu$ in the Lagrangian which are, up to total derivatives,

$$\begin{aligned}
& -\frac{1}{4}\, F_{\mu\nu}^2 - \frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 + \frac{1}{2}\,(qv)^2\, A_\mu^2 \\
& \qquad = \frac{1}{2}\, A^\mu\, [g_{\mu\nu}(\partial^2 + q^2 v^2) - (1 - \xi^{-1})\, \partial_\mu\partial_\nu\,]\, A^\nu\,,
\end{aligned} \tag{8.123}$$

from which the inverse propagator can be immediately read off:

$$[D_{\mu\nu}(p)]^{-1} = g_{\mu\nu}\,(-p^2 + M_{\rm A}^2) + (1 - \xi^{-1})\,p_\mu p_\nu\,, \tag{8.124}$$

or in terms of projection operators,

$$[D_{\mu\nu}(p)]^{-1} = -(p^2 - M_{\rm A}^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - \xi^{-1}(p^2 - \xi M_{\rm A}^2)\frac{p_\mu p_\nu}{p^2}\,.$$

The propagator itself is then

$$D_{\mu\nu}(p) = -(p^2 - M_{\rm A}^2 + {\rm i}\varepsilon)^{-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - \xi(p^2 - \xi M_{\rm A}^2)^{-1}\frac{p_\mu p_\nu}{p^2}\,,$$

which reduces to

$$D_{\mu\nu}(p) = \frac{-g_{\mu\nu} + (1-\xi)p_\mu p_\nu/(p^2 - \xi\,M_{\rm A}^2)}{p^2 - M_{\rm A}^2 + {\rm i}\varepsilon}\,. \tag{8.125}$$

Note that the poles at $p^2 = \xi M_{\rm A}^2$ in (121) and (125) are unphysical and need not be defined with an i$\varepsilon$ term. They will be canceled out in any physical transition amplitude (as shown by 't Hooft).

When $M_A = 0$, one recovers the propagator for the photon or gluon,

$$D_{\mu\nu}(p) = \frac{-g_{\mu\nu} + (1-\xi)p_\mu p_\nu/p^2}{p^2 + {\rm i}\varepsilon}\,. \tag{8.126}$$

Familiar gauges correspond to special values of $\xi$. With $\xi = 1$, we recover the Feynman gauge often used in QED,

$$D_{\mu\nu}(p) = \frac{-g_{\mu\nu}}{p^2 - M_A^2 + {\rm i}\varepsilon}\,, \tag{8.127}$$

while for $\xi = 0$, it is the Landau gauge, in which the propagator depends only on the transverse projection operator,

$$D_{\mu\nu}(p) = \frac{-g_{\mu\nu} + p_\mu p_\nu/p^2}{p^2 - M_{\rm A}^2 + {\rm i}\varepsilon}\,. \tag{8.128}$$

We can see that for any finite value of $\xi$, the propagator at large momentum behaves as $1/p^2$, just as in the massless case, and the corresponding gauge is manifestly renormalizable. But as $\xi \to \infty$, (125) tends to the ordinary propagator for a massive vector field

$$D_{\mu\nu}(p) = \frac{-g_{\mu\nu} + p_\mu p_\nu/M_A^2}{p^2 - M_A^2 + {\rm i}\varepsilon}\,, \tag{8.129}$$

while the propagator for the erstwhile Nambu–Goldstone mode in (121) tends to zero, suggesting that this field will drop out of the system. The gauge $\xi \to \infty$ coincides with the unitary gauge.

### 8.6.2 Non-Abelian Symmetry

We now generalize the above considerations to local gauge symmetry by introducing the general Yang–Mills fields to make the model of Sect. 8.5.4 invariant under a local gauge group G. We consider a real $n$-dimensional representation of G spanned by $n$-component scalar fields $\boldsymbol{\phi}$ and in which the transformations are generated by $N$ imaginary and antisymmetric $n \times n$ matrices $T_i$ for $i = 1, \ldots, N$. To each generator corresponds a vector gauge field, $A_{i\mu}$, so as to satisfy G gauge invariance (by which we mean gauge invariance under the symmetry group $G$ ). Thus, we consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(\boldsymbol{D}_\mu \boldsymbol{\phi})^{\mathrm{T}} \boldsymbol{D}^\mu \boldsymbol{\phi} - V(\boldsymbol{\phi}) - \frac{1}{4}\, F_{i\mu\nu} F_i^{\mu\nu}\,, \tag{8.130}$$

with the G-covariant derivative

$$\boldsymbol{D}_\mu \boldsymbol{\phi} = (\partial_\mu + \mathrm{i} g \boldsymbol{A}_\mu)\, \boldsymbol{\phi} = (\partial_\mu + \mathrm{i} g A_{j\mu} T_j)\, \boldsymbol{\phi}\,,$$

and the G-covariant field tensors associated with the gauge fields

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu - g\, f_{ijk}\, A_j^\mu A_k^\nu\,.$$

An explicit form of $V$ in terms of $\boldsymbol{\phi}$ is not essential, all we need is that it respects the G symmetry and develops degenerate minima at nonzero constant field values $|\boldsymbol{\phi}|^2 = v^2$. Then, the gauge invariance is spontaneously broken when the system arbitrarily selects for itself a vacuum state such that $\langle \boldsymbol{\phi} \rangle$ is some constant vector $\boldsymbol{v}$, which we call the *vacuum vector*, satisfying the condition that $\boldsymbol{v}^2$ minimizes $V$. Now we suppose that the symmetry breaking leaves the vacuum invariant under a subgroup S of G and that the generators $T_i$ of G are chosen so that $T_j$, for $j = 1, \ldots, M$ and $M < N$, generate S . Since by assumption $\boldsymbol{v}$ is invariant under S but noninvariant under its complement in G, we have

$$T_j \boldsymbol{v} = 0\,, \qquad j = 1, \ldots, M\,; \tag{8.131}$$

$$T_k \boldsymbol{v} \neq 0\,, \qquad k = M+1, \ldots, N\,. \tag{8.132}$$

Just as the algebra g of G is defined by all the linear combinations $\sum_{i=1}^N c_i T_i$, so too is the algebra $\mathrm{g_S}$ of the subgroup S defined by all the combinations $\sum_{j=1}^M c_j T_j$. Subalgebra $\mathrm{g_S}$ has dimension $M$; it annihilates the vacuum. The orthogonal complement to $\mathrm{g_S}$ in g has dimension $N - M$; its elements applied on the vacuum vector yield $N - M$-dimensional vectors, $\sum_{k=M+1}^N c_k T_k \boldsymbol{v}$, which span the space of the Nambu–Goldstone modes.

Taking advantage of the gauge invariance of (130), we may perform a gauge transformation on all the fields without changing the underlying physics. The particular gauge transformation $U$ is so chosen to cancel the quadratic coupling between the gauge fields and the scalar fields, which is equivalent to requiring

$$\big((U\boldsymbol{\phi})^{\mathrm{T}}(x)\,T_i\boldsymbol{v}\big) = 0 \qquad \text{for all } i \text{ and all } x. \tag{8.133}$$

That such a (unitary) gauge always exists was proved by Weinberg. Given this general result, let us define an $n$-component field $\boldsymbol{H}(x)$ orthogonal to all $T_k\boldsymbol{v}$ for $k = M+1,\ldots,N$. As $T_i$ are antisymmetric matrices (as in Sect. 8.5.4), we also have $(\boldsymbol{v}^{\mathrm{T}}\,T_i\boldsymbol{v}) = 0$, and therefore,

$$\big((\boldsymbol{v}+\boldsymbol{H})^{\mathrm{T}}\,T_k\boldsymbol{v}\big) = 0\,, \qquad k = M+1,\ldots,N\,, \tag{8.134}$$

in addition to the identity

$$\big((\boldsymbol{v}+\boldsymbol{H})^{\mathrm{T}}\,T_j\boldsymbol{v}\big) = 0\,, \qquad j = 1,\ldots,M\,, \tag{8.135}$$

which holds because of the invariance of the vacuum vector under S. Taken together, these relations yield

$$\big((\boldsymbol{v}+\boldsymbol{H})^{\mathrm{T}}\,T_i\boldsymbol{v}\big) = 0\,, \qquad i = 1,\ldots,N\,. \tag{8.136}$$

As we will now see, it proves useful to adopt a parameterization of fields similar in form to (101):

$$\boldsymbol{\phi} = \exp\left(\frac{\mathrm{i}}{v}\sum_{k=M+1}^{N}\xi_k T_k\right)(\boldsymbol{v}+\boldsymbol{H})\,. \tag{8.137}$$

In this parameterization, the independent fields are the $N-M$ would-be Nambu–Goldstone modes $\xi_k$, with $k = M+1,\ldots,N$, and the $n-N+M$ independent components of $\boldsymbol{H}$ representing the Higgs bosons. It now becomes clear that the transformation $U$ needed to satisfy (133) or (136) is

$$U = \exp\left(-\frac{\mathrm{i}}{v}\sum_{k=M+1}^{N}\xi_k T_k\right)\,. \tag{8.138}$$

Since the Lagrangian (130) is invariant under the local group G, it remains unchanged with the fields written in the new gauge

$$\begin{aligned}
\boldsymbol{\phi}' &= U\boldsymbol{\phi} = \boldsymbol{v}+\boldsymbol{H}\,,\\
\boldsymbol{A}'_\mu &= U\boldsymbol{A}_\mu U^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger\,,\\
\boldsymbol{F}'_{\mu\nu} &= U\boldsymbol{F}_{\mu\nu}U^\dagger\,.
\end{aligned} \tag{8.139}$$

Thus, in this U-gauge, the fields $\xi_k$ have completely vanished, and (130) involves only $\boldsymbol{H}$ and $A'_{j\mu}$, which we will now simply write $A_{j\mu}$,

$$\mathcal{L} = \frac{1}{2}\left(\boldsymbol{D}_\mu(\boldsymbol{v}+\boldsymbol{H})\right)^{\mathrm{T}} \boldsymbol{D}^\mu(\boldsymbol{v}+\boldsymbol{H}) - V(\boldsymbol{v}+\boldsymbol{H}) - \frac{1}{2}\partial_\mu A_{i\nu}\left(\partial^\mu A_i^\nu - \partial^\nu A_i^\mu\right). \tag{8.140}$$

The 'kinetic' part can be expanded as

$$\begin{aligned}
&\left(\boldsymbol{D}_\mu(\boldsymbol{v}+\boldsymbol{H})\right)^{\mathrm{T}} \boldsymbol{D}^\mu(\boldsymbol{v}+\boldsymbol{H}) \\
&\quad = \left(\partial_\mu \boldsymbol{H}^{\mathrm{T}} + \mathrm{i}g(\boldsymbol{v}+\boldsymbol{H})^{\mathrm{T}}\boldsymbol{A}_\mu^{\mathrm{T}}\right)\left(\partial^\mu \boldsymbol{H} + \mathrm{i}g\boldsymbol{A}^\mu(\boldsymbol{v}+\boldsymbol{H})\right) \\
&\quad = \partial_\mu \boldsymbol{H}^{\mathrm{T}}\,\partial^\mu \boldsymbol{H} + 2\mathrm{i}g\,\partial_\mu \boldsymbol{H}^{\mathrm{T}}\,\boldsymbol{A}^\mu(\boldsymbol{v}+\boldsymbol{H}) + g^2(\boldsymbol{v}+\boldsymbol{H})^{\mathrm{T}}\boldsymbol{A}_\mu\boldsymbol{A}^\mu\,(\boldsymbol{v}+\boldsymbol{H}).
\end{aligned} \tag{8.141}$$

We observe that the quadratic mixing term of $\boldsymbol{A}_\mu$ and $\boldsymbol{H}$ can now be disposed of, as expected from (136),

$$2\mathrm{i}g\,\partial_\mu \boldsymbol{H}^{\mathrm{T}}\,\boldsymbol{A}^\mu \boldsymbol{v} = 2\mathrm{i}g\,A_{i\mu}\,\partial^\mu \boldsymbol{H}^{\mathrm{T}} T_i \boldsymbol{v} = 0\,. \tag{8.142}$$

The Lagrangian then reduces to

$$\begin{aligned}
\mathcal{L} = &\frac{1}{2}\left(\partial_\mu \boldsymbol{H}^{\mathrm{T}}\,\partial^\mu \boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}\mathcal{M}_{\mathrm{B}}^2\boldsymbol{H}\right) \\
&- \frac{1}{2}\,\partial_\mu A_{i\nu}\left(\partial^\mu A_i^\nu - \partial^\nu A_i^\mu\right) + \frac{1}{2}g^2\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}_\mu\boldsymbol{A}^\mu\,\boldsymbol{v} \\
&+ \mathrm{i}\,g\,\partial_\mu \boldsymbol{H}^{\mathrm{T}}\,\boldsymbol{A}^\mu \boldsymbol{H} + g^2\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}_\mu\boldsymbol{A}^\mu\boldsymbol{H} + \frac{1}{2}\,g^2\boldsymbol{H}^{\mathrm{T}}\boldsymbol{A}_\mu\boldsymbol{A}^\mu\boldsymbol{H} + \dots\,,
\end{aligned} \tag{8.143}$$

where ... indicates the cubic and quartic self-coupling terms in $\boldsymbol{H}$, whose details depend on the assumed potential $V$. The surviving real scalar Higgs fields are massive, with squared masses determined by the matrix

$$\left(\mathcal{M}_{\mathrm{B}}^2\right)_{ab} = \left.\frac{\partial^2 V}{\partial\phi_a\,\partial\phi_b}\right|_{\phi=v}\,. \tag{8.144}$$

The squared-mass matrix for the vector mesons

$$g^2(\boldsymbol{v}^{\mathrm{T}} T_\ell T_k\,\boldsymbol{v}) \tag{8.145}$$

is real, symmetric, and positive-definite, and is nonvanishing for $\ell, k = \mathrm{M}{+}1, \dots, N$. Of all the gauge fields, only those associated with the symmetry-breaking part of G, that is, $A_k^\mu$ with $k = M+1,\dots,N$, acquire *masses* and *longitudinal components*, as shown by the presence of the inhomogeneous terms in the variations under the transformation (139):

$$\delta A_{k\mu} = \frac{1}{gv}\partial_\mu\xi_k - \frac{1}{v}f_{km\ell}A_{m\mu}\xi_\ell\,, \tag{8.146}$$

whereas the others, $A_j^\mu$ with $j = 1, \ldots, M$, associated with the surviving symmetry, remain *massless, transversely polarized*, and transform homogeneously under (139). The number of independent degrees of freedom remains the same before and after the symmetry breaking. The original $n$ real scalar fields and $N$ massless gauge mesons, producing altogether $n + 2N$ degrees of freedom, are replaced after the symmetry breaking by $n - N + M$ Higgs bosons, $M$ massless vector fields, and $N - M$ massive vector fields for a total of $n - N + M + 2M + 3(N - M) = n + 2N$ degrees of freedom.

As we have discussed above, a unitary gauge leads to a formalism that is simple to interpret, but has the disadvantage of not being manifestly renormalizable. It is more useful for practical calculations to adopt a manifestly renormalizable gauge so that the powerful techniques developed for renormalizable theories can be applied. This can be accomplished by a generalization of the gauge-fixing Lagrangian (118),

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A_i^\mu - \mathrm{i}g\xi\boldsymbol{\phi}^{\rm T}T_i\boldsymbol{v})^2\,, \tag{8.147}$$

designed so that the quadratic mixing terms between $A_{i\mu}$ and $\partial_\mu\boldsymbol{\phi}$ found here and in the Lagrangian (130) exactly cancel out.

We have limited ourselves in this chapter to a discussion of the mass generation of gauge bosons. When matter fields are introduced, they may not be allowed by gauge invariance to have explicit masses in the basic Lagrangian. It is possible however to induce their masses by coupling matter fields to the Higgs fields in a gauge-invariant way. We will study how this mechanism can make quarks and leptons massive in the context of the standard model of the electroweak interaction.

## Problems

**8.1 Equations of motion.** Show that the equations of motion corresponding to the Lagrangian (42) are

$$\begin{aligned}(\mathrm{i}\gamma^\mu\boldsymbol{D}_\mu - m)\,\psi &= 0\,,\\ \boldsymbol{D}^\mu F_{\mu\nu} &= g\,\overline{\psi}\gamma_\nu T_j\psi\,.\end{aligned}$$

**8.2 Group multiplication in gauge groups.** Show that, for any element $h$ of the gauge group, the transformation rule for the gauge field

$$h : \boldsymbol{A}_\mu \to \boldsymbol{A}'_\mu = U\boldsymbol{A}_\mu U^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger$$

satisfies the group multiplication law, that is, if $h : \boldsymbol{A}_\mu \to \boldsymbol{A}'_\mu$ and $h' : \boldsymbol{A}'_\mu \to \boldsymbol{A}''_\mu$ then $h'' : \boldsymbol{A}_\mu \to \boldsymbol{A}''_\mu$, where $h'' = h'h$.

**8.3 The linear $\sigma$–$\pi$ model.** Consider a model for a real field $\sigma$, transforming as an isosinglet and three real scalar fields $\phi_i$, forming an isotriplet.

Call the conjugate momenta of fields $\pi^i_\mu = \partial_\mu \phi_i$, $\pi^4_\mu = \partial_\mu \sigma$ and their time components $\pi^i_0 = \pi_i$ and $\pi^4_0 = \pi_4$. The fields are quantum operators satisfying the canonical commutation relations at equal times. The Lagrangian of the model is given by

$$\mathcal{L}_\mathrm{s} = \frac{1}{2}(\partial_\mu \phi_i \partial^\mu \phi^i + \partial_\mu \sigma \partial^\mu \sigma) - V(\boldsymbol{\phi}^2 + \sigma^2)\,. \tag{1}$$

(a) Show that the model is invariant under the following two global transformations of their internal degrees of freedom ($\sigma \to \sigma + \delta_i \sigma\,, \boldsymbol{\phi} \to \boldsymbol{\phi} + \delta_i \boldsymbol{\phi}$):
– isospin rotation:

$$\delta_i \sigma = 0\,; \quad \delta_i \phi_j = \epsilon_{ijk} \omega_i \phi_k\,, \text{ (no sum over } i). \tag{2}$$

– chiral transformation:

$$\delta_i \sigma = \omega_i \phi_i\,; \quad \delta_i \phi_j = -\delta_{ij} \omega_i \sigma\,, \text{ (no sum over } i). \tag{3}$$

(b) Show that the associated conserved isospin and axial currents are $V_{i\mu} = \epsilon_{ijk} \phi_j \pi^k_\mu$, and $A_{i\mu} = \pi^i_\mu \sigma - \pi^4_\mu \phi_i$; and that the corresponding conserved charges, $Q_i$ and $Q^5_i$ satisfy

$$[Q_i, Q_j] = \mathrm{i}\epsilon_{ijk} Q_k\,,$$
$$[Q_i, Q^5_j] = \mathrm{i}\epsilon_{ijk} Q^5_k\,,$$
$$[Q^5_i, Q^5_j] = \mathrm{i}\epsilon_{ijk} Q_k\,.$$

Show that $Q^+_i \equiv \frac{1}{2}(Q_i + Q^5_i)$ and $Q^-_i \equiv \frac{1}{2}(Q_i - Q^5_i)$ form two independent commuting SU(2) algebras, so that the algebra of the model is a semisimple algebra, $\mathrm{SU}(2) \times \mathrm{SU}(2)$.
(c) Assuming that $V = \frac{1}{2}\mu^2(\sigma^2 + \boldsymbol{\phi}^2) + \frac{1}{4}\lambda\,(\sigma^2 + \boldsymbol{\phi}^2)^2$, where $\lambda > 0$ and $\mu^2 < 0$. The potential $V$ has minima for fields satisfying $\sigma^2 + \boldsymbol{\phi}^2 = -\mu^2/\lambda$. Select the vacuum such that $\langle \phi_i \rangle = 0\,,\quad$ and $\quad\langle \sigma \rangle = v = \sqrt{-\mu^2/\lambda}$, thus provoking a symmetry breakdown. Now define $\sigma' = \sigma - v$. Calculate the masses of $\phi_i$ and $\sigma'$. Calculate the commutation relations of $Q_i$ and $Q^5_i$ with $\sigma'$ and $\phi_i$, and show that $Q^5_i$ generate a symmetry that is broken in the vacuum.
(d) To $\mathcal{L}_\mathrm{s}$, add the following Lagrangian for an isodoublet of fermions, interacting with $\sigma$ and $\phi_i$,

$$\mathcal{L}_\mathrm{F} = \overline{\psi}(\mathrm{i}\gamma \cdot \partial - m_0)\psi - g\,\overline{\psi}(\sigma + \mathrm{i}\tau_j \phi_j \gamma_5)\psi\,. \tag{4}$$

Together $\mathcal{L}_\mathrm{s}$ and $\mathcal{L}_\mathrm{F}$ define the Gell-Mann–Levy model. The isospin and chiral transformations (2) and (3) are supplemented by the following for $\psi$:

$$\delta_i \psi = \tfrac{1}{2}\mathrm{i}\omega_i \tau_i \psi\,, \quad \text{(no sum over } i).$$
$$\delta_i \psi = \tfrac{1}{2}\mathrm{i}\omega_i \tau_i \gamma_5 \psi\,, \quad \text{(no sum over } i).$$

Show that $\mathcal{L}_\mathrm{F}$ is invariant to isospin rotations for $m$ arbitrary, but is invariant to chiral transformation only for $m_0 = 0$. Assume now $m_0 = 0$. Consider the full system described by $\mathcal{L}_\mathrm{s} + \mathcal{L}_\mathrm{F}$, and show that when there is spontaneous symmetry breaking by (8), the fermion acquires a mass, $m = gv$. Express the parameters $\mu^2$, $\lambda$, and $v$ in terms of g, $m$, and $m_{\sigma'}$.

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