5 Discrete Symmetries

The group of all Lorentz transformations includes the proper continuous transformations already studied in previous chapters and the discrete transformations to be treated in this chapter. The latter class of transformations deals with *space* and *time inversions* as well as all operations formed by successive applications of a space or time inversion and a proper continuous transformation.

Invariance of physical systems with respect to the proper Lorentz group is one of the best-established properties, so much so that it is universally accepted as a fundamental principle of contemporary physics. It is then natural and, from the esthetic viewpoint, desirable to expect all physical phenomena to be invariant to the inversion operations as well: left–right symmetry and past–future symmetry. After all, the dynamic equations of classical mechanics appear unchanged in these transformations. What a surprise when it was discovered that the symmetry under space reflections was violated by the weak interactions. It then seems quite possible that the reversal of the time direction is not a universal symmetry either.

Related to these inversion operations is the *charge conjugation*, which acts not on space-time but rather on internal space. It reverses the signs of the electric charges of fields and all of their other additive quantum numbers (also called generalized charges) without changing any of their kinematic attributes, thus converting particles into antiparticles. There exists in fact a close relationship between these three discrete transformations: a successive application of all three transformations in any order constitutes a symmetry operation for all quantum field theories that satisfy very general conditions, even in cases where individual transformations may be violated in some interactions.

In this chapter we shall discuss applications of the inversion operations in quantum field theories. In view of model building, it is just as important to study the implications of invariance of physical systems to these transformations so as to discover how and in what circumstances these symmetries are violated.

5.1 Parity

The elementary discrete space transformation is the reflection in a spatial plane. However, as a reflection in a plane is equivalent to a rotation through an angle π about an axis perpendicular to that plane followed by an inversion with respect to the intersection of that axis with the plane, it suffices to consider without any loss of generality just the inversion. This operation,

$$\mathcal{P}:\ \boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{x}\,, \quad t \to t' = \ t\,, \tag{5.1}$$

is the basic *improper orthochronous* Lorentz transformation defined by

$$a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{5.2}$$

It is also often referred to as the *parity* operation. Invariance of a physical system to inversion means that the system cannot distinguish left from right in any interactions. On the other hand, the detection in the system of some physical quantity with a left–right asymmetry is a clear signal that the symmetry is broken. In the next few paragraphs, we will generalize the notion of space inversion of quantum mechanics to field theories, in particular defining the transformation rules for observables and introducing the concept of intrinsic parity, and briefly discuss the behavior of the fundamental interactions under the parity operation.

5.1.1 Parity in Quantum Mechanics

The inversion transforms the momentum $\boldsymbol{p}$ into $-\boldsymbol{p}$, as is evident from the operational form $\boldsymbol{p} = -\mathrm{i}\nabla$. The orbital angular momentum $\boldsymbol{L}$ remains unchanged since $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$. The generalized angular momentum $\boldsymbol{J}$ is also unchanged since space inversion commutes with all space rotations (see Fig. 5.1). A three-vector that changes sign in inversion (for example, $\boldsymbol{x}$ or $\boldsymbol{p}$) is called a *polar vector*; if it remains unchanged (for example, $\boldsymbol{x} \times \boldsymbol{p}$ or $\boldsymbol{J}$), it is called an *axial vector*. A scalar quantity that remains unchanged is a *scalar* (e.g. $\boldsymbol{p}^2$), but if it changes sign under inversion, it is a *pseudoscalar* (e.g. $\boldsymbol{p}\cdot\boldsymbol{J}$).

We assume there exists a linear operator $\mathcal{P}$ that performs space inversions on the Hilbert space and relates a given state vector to the transformed state vector. It is chosen to be *unitary* to preserve the normalization and orthogonality of states. Since $\mathcal{P}^2$, when acting on a state, brings it back to the original state, the phases can be fixed such that $\mathcal{P}^2 = 1$ provided we ignore (for the moment) spin degrees of freedom. With $\mathcal{P}^2 = 1$ and $\mathcal{P}^\dagger\mathcal{P} = 1$, the operator $\mathcal{P}$ is Hermitian ($\mathcal{P}^\dagger = \mathcal{P}^{-1} = \mathcal{P}$) and therefore is an *observable*.

If a system described by the Hamiltonian H is invariant to inversion,

$$\mathcal{P}H\mathcal{P}^{-1} = H \quad \text{or} \quad [\mathcal{P}, H] = 0\,, \tag{5.3}$$

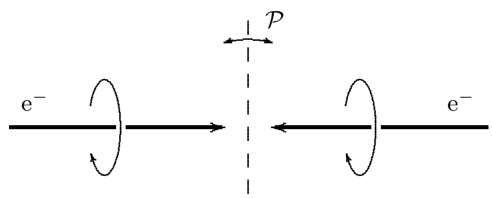

Fig. 5.1. $\mathcal{P}$ reverses the momentum of a particle without flipping its spin

$\mathcal{P}$ is a constant of the motion, and simultaneous eigenvectors of H and $\mathcal{P}$ can be found. The corresponding eigenvalue of $\mathcal{P}$ for the state is called the *parity* of the state, $\eta = +1$ or $\eta = -1$. Therefore, parity is a multiplicative quantum number; that is, the parity of a compound system is equal to the product of the parities of its individual components.

Consider for example a particle in an orbit of angular momentum ℓ. The angular part of its wave function is given by the spherical harmonics $Y_{\ell m}(\theta, \varphi)$. In inversion, $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$, and $Y_{\ell m}(\theta, \varphi)$ changes into $Y_{\ell m}(\pi - \theta, \varphi + \pi) = (-)^{\ell}\, Y_{\ell m}(\theta, \varphi)$. It follows that

$$\mathcal{P}\ |\ell m\rangle = (-)^{\ell}\ \ |\ell m\rangle\ . \tag{5.4}$$

Thus, an eigenvector of orbital angular momentum ℓ also has a well-defined parity, which is $(-1)^{\ell}$. We have also verified in passing that the parity and the orbital angular momentum are simultaneously good quantum numbers, i.e. $[\mathcal{P}, \boldsymbol{L}] = 0$. In contrast, since $\mathcal{P}\boldsymbol{p}\mathcal{P}^{\dagger} = -\boldsymbol{p}$, an eigenvector of momentum does not have a well-defined parity and, conversely, an eigenvector of parity does not have a well-defined momentum. The plane wave of a spinless particle, $\langle \boldsymbol{x}|\ \boldsymbol{p}\rangle = \exp(-\mathrm{i}Et + \mathrm{i}\boldsymbol{p}{\cdot}\boldsymbol{x})$, becomes after inversion a plane wave propagating in the reversed direction:

$$\langle \boldsymbol{x}'\ |\boldsymbol{p}\rangle = \langle \mathcal{P}\boldsymbol{x}\ |\boldsymbol{p}\rangle = \exp[-\mathrm{i}Et + \mathrm{i}(-\boldsymbol{p}){\cdot}\boldsymbol{x}\,]\,.$$

Let $\mathcal{O}_+$ be an *even* operator under inversion, $\mathcal{P}\mathcal{O}_+\mathcal{P}^{\dagger} = \mathcal{O}_+$, and $\mathcal{O}_-$ an *odd* operator, $\mathcal{P}\mathcal{O}_-\mathcal{P}^{\dagger} = -\mathcal{O}_-$. Their matrix elements between states of well-defined parities are given by

$$\begin{aligned}
\langle \eta'|\ \mathcal{O}_+\ |\eta''\rangle &= \langle \eta'|\ \mathcal{P}^{\dagger}\mathcal{P}\mathcal{O}_+\mathcal{P}^{\dagger}\mathcal{P}\ |\eta''\rangle = \ \eta'\eta''\,\langle \eta'|\ \mathcal{O}_+\ |\eta''\rangle\ ,\\
\langle \eta'|\ \mathcal{O}_-\ |\eta''\rangle &= \langle \eta'|\ \mathcal{P}^{\dagger}\mathcal{P}\mathcal{O}_-\mathcal{P}^{\dagger}\mathcal{P}\ |\eta''\rangle = -\eta'\eta''\,\langle \eta'|\ \mathcal{O}_-\ |\eta''\rangle\ .
\end{aligned} \tag{5.5}$$

These results show that an *even observable* has vanishing matrix elements between states of opposite parities, whereas an *odd observable* has vanishing matrix elements between states of equal parities. This selection rule is useful in studies of nuclear and electromagnetic transitions.

We have ignored up to now the notion of *intrinsic parity.* In fact, the parity of a state arises from both the relative motion of all the particles composing the system and the intrinsic parity of every particle. The intrinsic parity of a nuclear system can be inferred once we know the angular momentum couplings of individual particles and define the intrinsic parity of the nucleon. Take for example the deuteron, which is known to be mainly in a ${}^3S_1$ state. (In spectroscopic studies, states are often labeled by ${}^{2S+1}\ell_J$, where ℓ, S, and J denote the orbital angular momentum, intrinsic spin, and total angular momentum; $\ell = 0, 1, 2, \ldots$ are labeled by the letters S, P, $D, \ldots$. So ${}^3S_1$ means $\ell = 0$, $S = 1$, and $J = 1$.) In its center-of-mass, the deuteron has orbital angular momentum $\ell = 0$ and hence parity $\eta = +1$, provided the relative parity of the neutron and proton in their center-of-mass is defined as $+1$. If we then treat the deuteron as a particle, we may define its intrinsic parity to be $\eta_{\mathrm{d}} = +1$.

Consider now the π^--capture reaction by a deuteron d, $\pi^- + \mathrm{d} \to \mathrm{n} + \mathrm{n}$. The neutron and the meson are taken for now as elementary particles. If ℓ and ℓ' stand for the relative orbital angular momenta of the particles respectively in the initial and final states, then the assumed parity conservation implies

$$\eta_\pi \eta_{\mathrm{d}} \,(-)^{\ell} = \eta_{\mathrm{n}} \eta_{\mathrm{n}} \,(-)^{\ell'} \;=\; (-)^{\ell'} \,.$$

In the capture process, the meson π^- is slowed down and captured in an atomic s-state ($\ell = 0$) of the deuteron, which means that the parity of the initial state is simply η_π and the total angular momentum is that of the deuteron, $J_{\mathrm{i}} = 1$. The final total angular momentum is, by conservation, $J_{\mathrm{f}} = 1$, and so the final two-neutron state must be one of the four configurations allowed by the rules of angular momentum couplings, namely ${}^3S_1$, ${}^3P_1$, ${}^1P_1$, ${}^3D_1$. However, since the final state is composed of two identical fermions, it must be antisymmetric under a permutation of the two neutrons (that is, ℓ and S must be both even or both odd numbers), which rules out all possibilities except ${}^3P_1$, evidently of negative parity. Thus, conservation of parity requires the existence of an intrinsic parity for π^- of value $\eta_\pi = -1$.

5.1.2 Parity in Field Theories

We now proceed to define the parities of boson and fermion fields and of their associated Fock operators. We shall discover, in particular, that the relative parity of a conjugate boson–antiboson pair is positive while that of a conjugate fermion–antifermion pair is negative.

Scalar and Pseudoscalar Fields. Let $\phi(t, \boldsymbol{x})$ be the operator that represents a Bose field of spin 0, and $\phi^\dagger(t, \boldsymbol{x})$ its Hermitian conjugate. Their transformations under inversion are defined by

$$\begin{aligned} \mathcal{P}\,\phi(t, \boldsymbol{x})\,\mathcal{P}^{-1} &= \eta_{\mathrm{B}} \phi(t, -\boldsymbol{x}) \,, \\ \mathcal{P}\,\phi^\dagger(t, \boldsymbol{x})\,\mathcal{P}^{-1} &= \eta_{\mathrm{B}} \phi^\dagger(t, -\boldsymbol{x}) \,, \end{aligned} \tag{5.6}$$

where $\eta_{\mathrm{B}} = +1$ or -1. Although ϕ behaves as a scalar field under proper Lorentz transformations, the inversion operation differentiates a *scalar field* with parity $\eta_{\mathrm{B}} = +1$ from a *pseudoscalar field* with $\eta_{\mathrm{B}} = -1$. This quantum number is determined by experiment. Considered as elementary entities, the meson $\mathrm{f}_0(980\ \mathrm{MeV})$ is a scalar particle, and the mesons π^0, $\pi^{\pm}$, K^0, and $\mathrm{K}^{\pm}$ are pseudoscalar particles.

In order to study the transformation properties of Fock states, we substitute into (6) the expansion series (2.99) of ϕ in terms of the operators $a_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}$, recalling that $\mathcal{P}$, as a linear operator in the Hilbert space, does not act on c-number quantities. We thus have, on the one hand,

$$\mathcal{P}\,\phi(x)\,\mathcal{P}^{-1} = \sum_p C_{\boldsymbol{p}} \left[\mathcal{P}\,a_{\boldsymbol{p}}\,\mathcal{P}^{-1}\mathrm{e}^{-\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})} + \mathcal{P}\,b^\dagger_{\boldsymbol{p}}\,\mathcal{P}^{-1}\mathrm{e}^{\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\right] , \tag{5.7}$$

($C_{\boldsymbol{p}}$ being the usual normalization of the field), and on the other hand,

$$\begin{aligned}\eta_{\mathrm{B}}\,\phi(t,-\boldsymbol{x}) &= \eta_{\mathrm{B}} \sum_p C_{\boldsymbol{p}} \left[a_{\boldsymbol{p}}\,\mathrm{e}^{-\mathrm{i}(Et+\boldsymbol{p}\cdot\boldsymbol{x})} + b^\dagger_{\boldsymbol{p}}\,\mathrm{e}^{\mathrm{i}(Et+\boldsymbol{p}\cdot\boldsymbol{x})}\right] \\ &= \eta_{\mathrm{B}} \sum_p C_{\boldsymbol{p}} \left[a_{-\boldsymbol{p}}\,\mathrm{e}^{-\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})} + b^\dagger_{-\boldsymbol{p}}\,\mathrm{e}^{\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\right] .\end{aligned} \tag{5.8}$$

Together with similar relations for $\phi^\dagger$, one obtains the basic properties

$$\mathcal{P}\,a_{\boldsymbol{p}}\,\mathcal{P}^{-1} = \eta_{\mathrm{B}}\,a_{-\boldsymbol{p}} , \qquad \mathcal{P}\,a^\dagger_{\boldsymbol{p}}\,\mathcal{P}^{-1} = \eta_{\mathrm{B}}\,a^\dagger_{-\boldsymbol{p}} , \tag{5.9}$$

$$\mathcal{P}\,b_{\boldsymbol{p}}\,\mathcal{P}^{-1} = \eta_{\mathrm{B}}\,b_{-\boldsymbol{p}} , \qquad \mathcal{P}\,b^\dagger_{\boldsymbol{p}}\,\mathcal{P}^{-1} = \eta_{\mathrm{B}}\,b^\dagger_{-\boldsymbol{p}} . \tag{5.10}$$

The parity of a one-boson state of momentum $\boldsymbol{p}$ is therefore given by

$$\begin{aligned}\mathcal{P}\ |\boldsymbol{p}\rangle &= \mathcal{P}\,a^\dagger_{\boldsymbol{p}}\ |0\rangle = \mathcal{P}a^\dagger_{\boldsymbol{p}}\mathcal{P}^{-1}\mathcal{P}\ |0\rangle \\ &= \eta_{\mathrm{B}}a^\dagger_{-\boldsymbol{p}}\ |0\rangle = \eta_{\mathrm{B}}\ |{-\boldsymbol{p}}\rangle ,\end{aligned} \tag{5.11}$$

where the parity of the vacuum is fixed by convention, $\mathcal{P}\ |0\rangle = +\ |0\rangle$. This result (11) restates the simple fact that momentum changes sign under inversion and that a state of a free boson of well-defined momentum is not an eigenstate of $\mathcal{P}$, exactly as found in the first quantization formalism. However, in the rest frame of the particle, where $\boldsymbol{p} = 0$,

$$\mathcal{P}\ |\boldsymbol{p} = 0\rangle = \eta_{\mathrm{B}}\ |\boldsymbol{p} = 0\rangle ; \tag{5.12}$$

that is, $|\boldsymbol{p} = 0\rangle$ is an eigenstate of $\mathcal{P}$. A particle at rest has a well-defined parity which is by definition its *intrinsic parity*, $\eta_{\mathrm{B}} = +1$ for a scalar particle and $\eta_{\mathrm{B}} = -1$ for a pseudoscalar particle. A similar analysis, starting from (10), shows that the corresponding antiparticle in a state of equal orbital angular momentum has equal parity. Hence the general result: a boson and its conjugate antiboson have *equal intrinsic parities*. It follows, for instance, that a $\pi^+\pi^-$ system in relative orbital angular momentum ℓ has parity $(-)^\ell$.

From the field properties (6), the transformation rules for dynamical variables can be found. For example, the current density for a boson field,

$$j^\mu(x) = \mathrm{i}\,[\phi^\dagger(x)\partial^\mu\phi(x) - (\partial^\mu\phi^\dagger(x))\phi(x)]\,, \tag{5.13}$$

transforms according to (6) as

$$\mathcal{P}\,j^0(t,\boldsymbol{x})\,\mathcal{P}^{-1} = +j^0(t,-\boldsymbol{x})\,, \qquad \mathcal{P}\,j^i(t,\boldsymbol{x})\,\mathcal{P}^{-1} = -j^i(t,-\boldsymbol{x})\,. \tag{5.14}$$

These transformation laws state that j^0 behaves as a scalar field, and $\boldsymbol{j}$ as a polar vector field under space inversion. With the transformation matrix $a^\mu{}_\nu$ defined in (2), the above results can be expressed concisely:

$$\mathcal{P}\,j^\mu(t,\boldsymbol{x})\,\mathcal{P}^{-1} = a^\mu{}_\nu\, j^\nu(t,-\boldsymbol{x})\,. \tag{5.15}$$

Electromagnetic Field. As the electromagnetic field is a Lorentz vector, one expects that

$$\mathcal{P}\,A^\mu(t,\boldsymbol{x})\,\mathcal{P}^{-1} = \eta_A\, a^\mu{}_\nu\, A^\nu(t,-\boldsymbol{x})\,. \tag{5.16}$$

Given (15) and the experimental observation that the electromagnetic interaction, $H_{\rm em} = qj^\mu A_\mu$, is invariant to space inversion, one may infer the value of the phase factor $\eta_A = +1$. In particular, the space components of the field transform according to

$$\mathcal{P}\,\boldsymbol{A}(t,\boldsymbol{x})\,\mathcal{P}^{-1} = -\boldsymbol{A}(t,-\boldsymbol{x})\,. \tag{5.17}$$

The transformation properties of the electromagnetic field operators in Fock space can be obtained by substituting into (17) the expansion series of the transverse $\boldsymbol{A}$ given in (2.156). Thus, the right-hand side of (17) reads

$$\begin{aligned} -\boldsymbol{A}(t,-\boldsymbol{x}) &= -\sum_{k\lambda} C_{\boldsymbol{k}}\left[\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\,a(\boldsymbol{k},\lambda)\mathrm{e}^{-\mathrm{i}\omega t-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{\epsilon}^*(\boldsymbol{k},\lambda)\,a^\dagger(\boldsymbol{k},\lambda)\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\ &= -\sum_{k\lambda} C_{\boldsymbol{k}}\left[\boldsymbol{\epsilon}(-\boldsymbol{k},\lambda)\,a(-\boldsymbol{k},\lambda)\mathrm{e}^{-\mathrm{i}k\cdot x} + \boldsymbol{\epsilon}^*(-\boldsymbol{k},\lambda)\,a^\dagger(-\boldsymbol{k},\lambda)\mathrm{e}^{\mathrm{i}k\cdot x}\right]. \end{aligned}$$

If the z axis is chosen to coincide with the propagation vector $\boldsymbol{k}$, the polarization vectors given in (2.153) become $\boldsymbol{\epsilon}(\hat{\boldsymbol{z}},\pm) = \mp\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})$. A rotation through 180° about the y axis brings $\hat{\boldsymbol{x}}$ to $-\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{z}}$ to $-\hat{\boldsymbol{z}}$, and the polarization vectors to

$$\boldsymbol{\epsilon}(-\hat{\boldsymbol{z}},\pm) = \pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} \mp \mathrm{i}\hat{\boldsymbol{y}}) = \boldsymbol{\epsilon}(\hat{\boldsymbol{z}},\mp) = -\boldsymbol{\epsilon}^*(\hat{\boldsymbol{z}},\pm)\,, \tag{5.18}$$

or more generally, $\boldsymbol{\epsilon}(\hat{\boldsymbol{k}}, \lambda) = \boldsymbol{\epsilon}(-\hat{\boldsymbol{k}}, -\lambda) = -\boldsymbol{\epsilon}^*(-\hat{\boldsymbol{k}}, \lambda)$. With this property taken into account, (17) becomes

$$\begin{aligned}\mathcal{P}\, \boldsymbol{A}(x)\, \mathcal{P}^{-1} = -\sum_{k\lambda} C_{\boldsymbol{k}} \Big[&\boldsymbol{\epsilon}(\boldsymbol{k}, \lambda)\, a(-\boldsymbol{k}, -\lambda)\, \mathrm{e}^{-\mathrm{i}k\cdot x} \\ &+ \boldsymbol{\epsilon}^*(\boldsymbol{k}, \lambda)\, a^\dagger(-\boldsymbol{k}, -\lambda)\, \mathrm{e}^{\mathrm{i}k\cdot x} \Big] , \end{aligned} \tag{5.19}$$

which implies the basic transformation rule for the photon Fock operator

$$\mathcal{P}\, a(\boldsymbol{k}, \lambda)\, \mathcal{P}^{-1} = -a(-\boldsymbol{k}, -\lambda)\, . \tag{5.20}$$

Thus, the photon has a *negative* intrinsic parity, $\eta_\gamma = -1$. Both its momentum and helicity change signs under the parity operation.

Dirac Fermion Field. Covariance of the Dirac equation requires the Dirac wave function $\psi(x)$ to transform as

$$\begin{aligned}\mathcal{P}\ :\ \ \psi(x) \to \psi'(t, -\boldsymbol{x}) &= S(a)\psi(x) \\ \text{or} \qquad \psi'(x) &= S(a)\psi(t, -\boldsymbol{x})\ .\end{aligned}$$

Here, $S(a)$ is defined by (3.13) and, with a as in (2), it becomes

$$S(a) = \eta_{\mathrm{F}}\, \gamma_0, \qquad \eta_{\mathrm{F}} = \pm 1\, , \tag{5.21}$$

which holds for any representation of γ_0. Here, η_{F} is the intrinsic parity of the Dirac particle to be determined by experiment.

In analogy with the classical wave function, the Dirac field operator transforms according to

$$\mathcal{P}\, \psi(x)\, \mathcal{P}^{-1} = \eta_{\mathrm{F}}\, \gamma_0\, \psi(t, -\boldsymbol{x})\, . \tag{5.22}$$

We first recall the expansion series for ψ given in (3.91):

$$\psi(x) = \sum_{\boldsymbol{p},s} C_{\boldsymbol{p}} \left[b(\boldsymbol{p}, s)u(\boldsymbol{p}, s)\, \mathrm{e}^{-\mathrm{i}p\cdot x} + d^\dagger(\boldsymbol{p}, s)v(\boldsymbol{p}, s)\, \mathrm{e}^{\mathrm{i}p\cdot x} \right]\, , \tag{5.23}$$

and also note that the free-particle spinors have the following properties, which can be proved by using their explicit expressions (3.45) and (3.46),

$$\begin{aligned}\gamma_0 u(-\boldsymbol{p}, s) &= \ u(\boldsymbol{p}, s)\, , \\ \gamma_0 v(-\boldsymbol{p}, s) &= -v(\boldsymbol{p}, s)\, .\end{aligned}$$

Then the right-hand side of (22) may be written as

$$\eta_{\mathrm{F}}\gamma_0\psi(t, -\boldsymbol{x}) = \eta_{\mathrm{F}} \sum_{\boldsymbol{p},s} C_{\boldsymbol{p}} \left[b(-\boldsymbol{p}, s)u(\boldsymbol{p}, s)\, \mathrm{e}^{-\mathrm{i}p\cdot x}\ -\ d^\dagger(-\boldsymbol{p}, s)v(\boldsymbol{p}, s)\, \mathrm{e}^{\mathrm{i}p\cdot x} \right]\, ,$$

which leads to the transformation properties of the Fock operators

$$\mathcal{P}\, b(\boldsymbol{p},s)\,\mathcal{P}^{-1} = \ \eta_{\rm F}\, b(-\boldsymbol{p},s), \qquad \mathcal{P}\, b^\dagger(\boldsymbol{p},s)\,\mathcal{P}^{-1} = \ \eta_{\rm F}\, b^\dagger(-\boldsymbol{p},s), \quad (5.24)$$

$$\mathcal{P}\, d(\boldsymbol{p},s)\,\mathcal{P}^{-1} = -\eta_{\rm F}\, d(-\boldsymbol{p},s), \qquad \mathcal{P}\, d^\dagger(\boldsymbol{p},s)\,\mathcal{P}^{-1} = -\eta_{\rm F}\, d^\dagger(-\boldsymbol{p},s)\,. \quad (5.25)$$

Thus, the one-fermion state $b^\dagger(\boldsymbol{p},s)\;|0\rangle$ transforms into $b^\dagger(-\boldsymbol{p},s)\;|0\rangle$, and the one-antifermion state $d^\dagger(\boldsymbol{p},s)\;|0\rangle$ into $-d^\dagger(-\boldsymbol{p},s)\;|0\rangle$, with their spin orientations unchanged. However, since momentum reverses direction, $\boldsymbol{J}{\cdot}\hat{\boldsymbol{p}}$ changes sign, and helicity states are not invariant to $\mathcal{P}$.

The negative sign on the right-hand sides of (25) means that the intrinsic parity of an antifermion is opposite in sign to that of the corresponding fermion. As a result, the parity of an electron–positron system in a relative s-state ($\ell = 0$) is necessarily odd:

$$\mathcal{P}b^\dagger(\mathbf{0},s)d^\dagger(\mathbf{0},s')\;|0\rangle = -b^\dagger(\mathbf{0},s)d^\dagger(\mathbf{0},s')\;|0\rangle \quad (5.26)$$

(to be contrasted with a boson–antiboson pair, such as $\pi^+\pi^-$). Thus, in general, the relative intrinsic parity is *even* for a self-conjugate boson–antiboson system and *negative* for a self-conjugate fermion–antifermion system.

Let us finally remark that the transformation rule (15) is also valid for the current density $j_\mu(x)$ of a Dirac field, as can be seen by applying the rules $\psi \to \eta\gamma_0\psi$ and $\bar\psi \to \eta\overline{\psi}\gamma_0$ to the expression $\overline{\psi}\gamma_\mu\psi$. More generally, a bilinear covariant transforms according to

$$\mathcal{P}\;\overline{\psi}(x)\Gamma\psi(x)\,\mathcal{P}^{-1} = \overline{\psi}(t,-\boldsymbol{x})\,\gamma_0\Gamma\gamma_0\;\psi(t,-\boldsymbol{x})\,. \quad (5.27)$$

For pseudoscalar, vector, and axial-vector operators, one needs

$$\begin{aligned}
\gamma_0\gamma_5\gamma_0 &= -\gamma_5; \\
\gamma_0\gamma_\mu\gamma_0 &= \begin{cases} +\gamma_\mu, & \text{if } \mu = 0; \\ -\gamma_\mu, & \text{if } \mu = 1,\,2,\,3; \end{cases} \\
\gamma_0\gamma_\mu\gamma_5\gamma_0 &= \begin{cases} -\gamma_\mu\gamma_5, & \text{if } \mu = 0; \\ +\gamma_\mu\gamma_5, & \text{if } \mu = 1,\,2,\,3. \end{cases}
\end{aligned}$$

5.1.3 Parity and Interactions

In the mid-1950s, it was discovered that while parity was conserved to a high degree of precision in strong and electromagnetic interactions, it was badly broken in weak interactions. Experiments were devised and carried out to map these irregularities, which have eventually led to a deeper understanding of the dynamics of particles.

Intrinsic Parity. If we set the phase of the vacuum state of some Hilbert space to 1, the absolute phase of any state vector in the space is defined as its phase relative to the vacuum. In discrete symmetries, there always exists an ambiguity in defining the phases of transformed states. Take for example a state of electric charge $|q\rangle$ also assumed to have good parity, $\mathcal{P}\,|q\rangle = \eta\,|q\rangle$. If now the parity operator is redefined as $\mathcal{P}' \equiv \mathcal{P}\exp(\mathrm{i}\alpha Q)$, where α is some real constant and Q the charge operator, then the parity of the state becomes $\eta' = \eta\exp(\mathrm{i}\alpha q)$ without causing observable physical effects on the system. Thus parity is defined only up to a phase factor. Its definition becomes unambiguous only if the charge vanishes, or is equal to the vacuum charge. More generally, the absolute parity is well defined only for *completely neutral particles* – particles that have all their generalized charges identically equal to zeros, such as the photon or the π^0 meson.

As we have seen, invariance of the electromagnetic interaction to inversion implies that the photon is *odd*, i.e. $\eta_\gamma = -1$. The parity of π^0 is also *negative*, a result that can be inferred from the following arguments. The meson π^0 has mean lifetime $\tau = 8 \times 10^{-17}\,$s, and decays in 99% of all cases through the channel $\pi^0 \to 2\gamma$. In the meson rest frame the initial angular momentum is $J_\mathrm{i} = 0$. By conservation, the final angular momentum is also $J_\mathrm{f} = 0$. The wave function of the two photons in the final state must contain the polarization vectors $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ and the relative momentum $\boldsymbol{k}$, which obey the transversality conditions $\boldsymbol{k}\cdot\boldsymbol{\epsilon}_1 = \boldsymbol{k}\cdot\boldsymbol{\epsilon}_2 = 0$. It must be a scalar function ($J_\mathrm{f} = 0$), linear in $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, and symmetric under permutation of the two photons, i.e. in the simultaneous exchanges $\boldsymbol{\epsilon}_1 \leftrightarrow \boldsymbol{\epsilon}_2$ and $\boldsymbol{k} \leftrightarrow -\boldsymbol{k}$. There are two possibilities consistent with these conditions:

(i) $\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2$, even under inversion, $\eta = +1$, and
(ii) $\boldsymbol{k}\cdot(\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2)$, odd under inversion, $\eta = -1$.

With φ denoting the angle between $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, the corresponding angular distributions are

(i) $|\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2|^2 \propto \cos^2\varphi\,, \qquad \eta = +1\,,$
(ii) $|\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2|^2 \propto \sin^2\varphi\,, \quad \eta = -1\,.$

Parity conservation says that the intrinsic parity of π^0 must be equal to $\eta_{\pi^0} = \eta\,\eta_\gamma^2 = \eta$. To determine η_{π^0}, it suffices to measure the photon polarizations in the final state. If the photons are found with predominant parallel linear polarizations ($\varphi = 0$), π^0 is a scalar particle; if on the contrary they are seen emitted with perpendicular polarizations ($\varphi = \pi/2$), π^0 is a pseudoscalar meson. Experiments show a clear preference for the second possibility: π^0 is a pseudoscalar meson with $\eta_\pi = -1$.

As for particles having nonvanishing additive quantum numbers, it is necessary to fix first the parities of a minimum number of reference particles. The relative parities of all other particles, whenever they can be defined, are determined from arguments based on parity conservation in parity-conserving reactions. Thus, one must define *at least* the parities of the neutron, of the proton (for processes that conserve electric charge and baryon number), and of Λ^0 (for reactions that conserve *strangeness*, a number characteristic of a class of unstable particles). The conventional choice is

$$\eta_\mathrm{n} = \eta_\mathrm{p} = \eta_\Lambda = +1\,. \tag{5.28}$$

Tests of Parity. For *electromagnetic interactions*, a category of tests of parity conservation consists in detecting transitions forbidden by the symmetry. Such tests can be made relatively simpler by concentrating on atomic states where the stronger hadronic effects are absent. For example, transitions between two atomic states of equal spins and equal parities, $J_{\rm i}^P = 1^+ \to J_{\rm f}^P = 1^+$, may proceed via the electric quadrupole and magnetic dipole modes (described by even parity operators in both cases), but are forbidden for the (odd parity) electric dipole mode, which would otherwise be the kinematically favored mode. The fact that transitions between these two states have not been observed indicates that if parity is broken at all in electromagnetic interactions, such a symmetry violation must be a very small effect.

Parity conservation in *strong interactions* can be similarly verified. A typical experiment consists in observing the α-decay of ^{20}Ne through a channel forbidden by conservation of parity, namely $J_{\rm i} = 1^+ \xrightarrow{\alpha} J_{\rm f} = 0^+$. The measured branching ratio for this mode is very small, again indicating that parity is indeed a symmetry of the strong interaction.

Parity conservation in a system demands its Lagrangian to obey

$$\mathcal{P}\,\mathcal{L}(t, \boldsymbol{x})\,\mathcal{P}^{-1} = \mathcal{L}(t, -\boldsymbol{x})\,. \tag{5.29}$$

As already mentioned, the electromagnetic interaction of a Dirac particle with an electromagnetic field obtained via the traditional *minimal coupling*, obtained by making the substitution $\mathrm{i}\partial_\mu \to \mathrm{i}\partial_\mu - qA_\mu$ (q being the particle charge) in the particle kinetic term,

$$q\bar{\psi}(x)\gamma_\mu\psi(x)\,A^\mu(x)\,, \tag{5.30}$$

is clearly parity conserving. For the strong couplings of fermions to mesons, two possibilities consistent with (29) are $g_1\,\bar{\psi}(x)\psi(x)\,\varphi(x)$ for a scalar meson φ and $\mathrm{i}g_2\,\bar{\psi}(x)\gamma_5\psi(x)\,\phi(x)$ for a pseudoscalar meson ϕ. Here g_1 and g_2 stand for dimensionless coupling constants.

In the mid-1950s, there was a persistent problem referred to as the τ–θ puzzle, that resisted any satisfactory solution for a long time. The so-called τ and θ particles have equal masses (494 MeV) and equal mean lifetimes (1.23×10^{-8} s), but decay through channels of opposite parities:

$$\begin{aligned} &\theta^+ \to \pi^+ + \pi^0\,, \\ &\tau^+ \to \pi^+ + \pi^+ + \pi^-. \end{aligned}$$

The θ mode is observed in 21% and the τ mode in 6% of all disintegrations. The values of their masses and lifetimes being identical, it is plausible that τ and θ are different decay modes of the same particle. However, this seemingly natural explanation has but one difficulty in that it runs counter to the accepted tenets of the time. *If parity is a conserved quantum number* in these decay processes, then, as τ and θ have opposite parities, they must be different particles in spite of their identical masses and mean lifetimes. On the other hand, *if parity is not a conserved quantum number*, the above argument does not hold and a given particle may decay through nonconserving interactions into two or three pions. Lee and Yang (1956) systematically re-examined the whole question and came to the conclusion that while parity was conserved in hadronic and electromagnetic interactions, there existed no firm experimental data verifying the validity of this symmetry in weak interactions. They suggested several ways to check the conservation or violation of parity in weak interactions. A series of experiments were subsequently performed and proved that the weak interactions indeed broke the parity symmetry. In particular, it was showed that τ and θ were in fact different manifestations of the same particle, now called the K meson.

The first observed weak process was the β-decay of neutron-rich nuclei, in which a bound neutron disintegrates into a proton, an electron and an antineutrino. The same process also occurs with free neutrons. E. Fermi described β-decay by a local interaction involving the four fermions, which was later generalized to the form

$$\mathcal{H}_\beta = \sum_i C_i \,\left(\overline{\psi}_{\mathrm{p}}\Gamma_i\psi_{\mathrm{n}}\right)\,\left(\overline{\psi}_{\mathrm{e}}\Gamma^i\psi_\nu\right)\ . \tag{5.31}$$

Here $C_i = C_{\mathrm{S}}, C_{\mathrm{V}}, C_{\mathrm{T}}, C_{\mathrm{A}}, C_{\mathrm{P}}$ are real or complex coupling constants of dimensions $[\mathrm{mass}]^{-2}$, and the matrices

$$\begin{aligned}\Gamma^i &= 1,\ \gamma^\mu,\ \sigma^{\mu\nu}/\sqrt{2},\ \gamma^\mu\gamma_5,\ \mathrm{i}\gamma_5\,,\\ \Gamma_i &= 1,\ \gamma_\mu,\ \sigma_{\mu\nu}/\sqrt{2},\ \gamma_\mu\gamma_5,\ -\mathrm{i}\gamma_5\end{aligned}$$

represent all possible couplings. If parity is *not* a symmetry, an even more general expression may be postulated:

$$\mathcal{H}_\beta = \sum_i C_i\,(\overline{\psi}_{\mathrm{p}}\Gamma_i\psi_{\mathrm{n}})\,[\overline{\psi}_{\mathrm{e}}\,(1+\alpha_i\gamma_5)\,\Gamma^i\psi_\nu]\,, \tag{5.32}$$

where α_i are dimensionless complex constants. Since the couplings $\overline{\psi}_{\mathrm{e}}\,\Gamma^i\psi_\nu$ and $\overline{\psi}_{\mathrm{e}}\gamma_5\,\Gamma^i\psi_\nu$ have opposite parities (cf. Table 3.1 or Table 5.3) the presence of both terms in the interaction breaks parity. If the momentum dependence in the neutron and proton spinors is neglected, the transition amplitude obtained from $\mathcal{H}_\beta$ can be written more simply in terms of the Pauli spinors:

$$\begin{aligned}\mathcal{M} \approx &(\chi_{\mathrm{p}}^\dagger\chi_{\mathrm{n}})\Big[C_{\mathrm{S}}\bar{u}_{\mathrm{e}}(p_{\mathrm{e}})(1+\alpha_{\mathrm{S}}\gamma_5)v_{\bar{\nu}}(p_\nu) + C_{\mathrm{V}}\bar{u}_{\mathrm{e}}(p_{\mathrm{e}})(1+\alpha_{\mathrm{V}}\gamma_5)\gamma^0 v_{\bar{\nu}}(p_\nu)\Big]\\ &+ (\chi_{\mathrm{p}}^\dagger\boldsymbol{\sigma}\chi_{\mathrm{n}})\boldsymbol{\cdot}\Big[C_{\mathrm{T}}\bar{u}_{\mathrm{e}}(p_{\mathrm{e}})(1+\alpha_{\mathrm{T}}\gamma_5)\boldsymbol{\sigma}\,v_{\bar{\nu}}(p_\nu)\\ &+ C_{\mathrm{A}}\bar{u}_{\mathrm{e}}(p_{\mathrm{e}})(1+\alpha_{\mathrm{A}}\gamma_5)\gamma_5\boldsymbol{\gamma}v_{\bar{\nu}}(p_\nu)\Big]\,.\end{aligned} \tag{5.33}$$

The S and V terms on the first line are responsible for the allowed Fermi transitions in nuclei, while the A and T terms produce the allowed Gamow–Teller transitions. All the constants C_i and α_i have been measured.

The constants α_i are first determined by measuring the longitudinal polarization of the emitted electron. A method consists in measuring the left–right asymmetry of atomic scattering of the electrons emitted in β-decay by observing the helicities of the electrons (or positrons) emitted in Fermi or Gamow–Teller nuclear transitions, in the decay of free neutrons, in inverse β-decay ($\mathrm{p} \to \mathrm{n e}^+ \nu_\mathrm{e}$), or in muon decay ($\mu^\pm \to \mathrm{e}^\pm \nu \bar{\nu}$). The results conclusively demonstrate that there exists a clear left–right asymmetry and thus confirm there is parity violation. Electrons emitted in β-decay are polarized in the direction opposite to their motion, whereas positrons are polarized in the direction of their motion:

$$\begin{aligned} &\left\langle \mathrm{e}^- \,|\, \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}} \,|\, \mathrm{e}^- \right\rangle = -\,v, \quad \text{electrons}\ , \\ &\left\langle \mathrm{e}^+ \,|\, \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}} \,|\, \mathrm{e}^+ \right\rangle = +\,v, \quad \text{positrons}\ . \end{aligned} \tag{5.34}$$

In the ultra-relativistic limit where velocity $v \to c = 1$, the helicities of emitted electrons and positrons go to -1 and $+1$ respectively. In the same limit, the projection for a left-handed electron becomes $(1-\gamma_5)/2$, and the interaction terms must involve only the electron left chiral components. In order to reproduce this limiting result, all constants α_i must be real and identical to $+1$, and (33) becomes

$$\begin{aligned} \mathcal{M} \approx (\chi_\mathrm{p}^\dagger \chi_\mathrm{n}) &\Big[C_\mathrm{S} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5) v_{\bar{\nu}} + C_\mathrm{V} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5)\gamma^0 v_{\bar{\nu}} \Big] \\ &+ (\chi_\mathrm{p}^\dagger \boldsymbol{\sigma} \chi_\mathrm{n}) \cdot \Big[C_\mathrm{T} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5)\boldsymbol{\sigma}\, v_{\bar{\nu}} + C_\mathrm{A} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5)\gamma_5 \boldsymbol{\gamma} v_{\bar{\nu}} \Big]\ . \end{aligned} \tag{5.35}$$

It remains to determine C_i. In transitions $J_\mathrm{i}^P = 0^+ \to J_\mathrm{f}^P = 0^+$, as in β^+-decay ${}^{14}\mathrm{O}(0^+) \to {}^{14}\mathrm{N}^*(0^+,\ 2.31\mathrm{MeV})$, only the Fermi couplings C_S and C_V are allowed since $\chi_\mathrm{p}^\dagger \boldsymbol{\sigma} \chi_\mathrm{n} = 0$. (Here χ_p and χ_n denote the Pauli spinors of bound nucleons.) In the angular distribution of the antineutrinos relative to the electron direction, the contributions from the scalar S coupling vary as $(1 - v_\mathrm{e}\cos\theta)$, and those from the vector V term as $(1 + v_\mathrm{e}\cos\theta)$. Comparisons with experimental observations confirm that Fermi transitions are of the V type, i.e. $C_\mathrm{S} = 0$. A similar analysis for Gamow–Teller transitions, as in β^--decays ${}^6\mathrm{He}(0^+) \to {}^6\mathrm{Li}(1^+)$ or ${}^{60}\mathrm{Co}(5^+) \to {}^{60}\mathrm{Ni}^*(4^+,\ 2.51\,\mathrm{MeV})$, shows that the angular distribution of the antineutrinos will be proportional to $(1 - {}^1\!/_3\, v_\mathrm{e}\cos\theta)$ for an axial-vector coupling and to $(1 + {}^1\!/_3\, v_\mathrm{e}\cos\theta)$ for a tensor coupling. Experiments are consistent with a small tensor coupling, showing that $C_\mathrm{T} \ll C_\mathrm{A}$.

Therefore, the amplitude for β-transitions is

$$\begin{aligned} \mathcal{M} &= (\chi_\mathrm{p}^\dagger \chi_\mathrm{n})\, C_\mathrm{V} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5)\gamma^0 v_{\bar{\nu}} + (\chi_\mathrm{p}^\dagger \boldsymbol{\sigma} \chi_\mathrm{n}) \cdot C_\mathrm{A} \bar{u}_\mathrm{e}(p_\mathrm{e})(1+\gamma_5)\gamma_5 \boldsymbol{\gamma} v_{\bar{\nu}} \\ &= (\chi_\mathrm{p}^\dagger \chi_\mathrm{n})\, C_\mathrm{V} \bar{u}_\mathrm{e}(p_\mathrm{e})\gamma^0(1-\gamma_5) v_{\bar{\nu}} + (\chi_\mathrm{p}^\dagger \boldsymbol{\sigma} \chi_\mathrm{n}) \cdot C_\mathrm{A} \bar{u}_\mathrm{e}(p_\mathrm{e})\boldsymbol{\gamma}(1-\gamma_5) v_{\bar{\nu}}\ . \end{aligned}$$

The magnitudes of the remaining coupling constants, C_V and C_A, are determined by the β-decay rates of the neutron and the pure Fermi transition in

^{14}O. Their relative phase is inferred from the electron angular distributions relative to the neutron spin in β-decay of polarized neutrons. This leads to

$$
\begin{aligned}
G_{\mathrm{F}} &\equiv \sqrt{2}\,C_{\mathrm{V}} = (1.14730 \pm 0.0006)\,10^{-5}\,\mathrm{GeV}^{-2}\,,\\
\alpha &\equiv \frac{C_{\mathrm{A}}}{C_{\mathrm{V}}} = (1.2573 \pm 0.0028)\,.
\end{aligned}
\tag{5.36}
$$

In summary, the nucleon β-decay may be described by the Lagrangian

$$
\begin{aligned}
\mathcal{L}_\beta(x) &= -\frac{G_{\mathrm{F}}}{\sqrt{2}}\,J_\mu(x)j^\mu(x) + \mathrm{h.c.}\\
&= -\frac{G_{\mathrm{F}}}{\sqrt{2}}\Big\{\big[\bar\psi_{\mathrm{p}}(x)\gamma_\mu(1-\alpha\gamma_5)\psi_{\mathrm{n}}(x)\big]\ \big[\bar\psi_{\mathrm{e}}(x)\gamma^\mu(1-\gamma_5)\psi_\nu(x)\big]\\
&\quad + \big[\bar\psi_{\mathrm{n}}(x)\gamma_\mu(1-\alpha\gamma_5)\psi_{\mathrm{p}}(x)\big]\ \big[\bar\psi_\nu(x)\gamma^\mu(1-\gamma_5)\psi_{\mathrm{e}}(x)\big]\Big\}\,.
\end{aligned}
\tag{5.37}
$$

The first term on the right-hand side describes the β-decay itself, in which a *right-handed antineutrino* is emitted. Its Hermitian conjugate (h.c.), given in the second term, makes the whole Lagrangian Hermitian; it represents the inverse β-decay processes, $\bar{\mathrm{n}} \to \bar{\mathrm{p}} + \mathrm{e}^+ + \nu$ or $\mathrm{p} \to \mathrm{n} + \mathrm{e}^+ + \nu$, in which a *left-handed neutrino* appears. Thus, this weak interaction involves only left-handed leptons and right-handed antileptons.

5.2 Time Inversion

The time inversion operator $\mathcal{T}$ reverses the sign of the time parameter,

$$
\mathcal{T}\ :\ x = (t, \boldsymbol{x}) \ \to\ x' = (-t, \boldsymbol{x})\,,
\tag{5.38}
$$

and changes physical variables accordingly,

$$
\begin{aligned}
\boldsymbol{p} &= m\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \ \to \boldsymbol{p}' = -\boldsymbol{p}\,,\\
\boldsymbol{L} &= \boldsymbol{x}\times\boldsymbol{p} \ \to \boldsymbol{L}' = -\boldsymbol{L}
\end{aligned}
$$

(see Fig. 5.2). Newton's equation for a particle acted on by a nondissipative and time-independent force is invariant to this transformation, so that if $\boldsymbol{x}(t)$ is an allowed trajectory, then $\boldsymbol{x}(-t)$ is equally allowed. Classical mechanics cannot determine the time arrow. On the other hand, assuming classical electromagnetism to be invariant as well, one sees that the electric and magnetic fields transform as $\boldsymbol{E} \to +\boldsymbol{E}$ and $\boldsymbol{B} \to -\boldsymbol{B}$, since the electric charge is unchanged but the electric current (the product of charge and velocity) changes sign under time inversion.

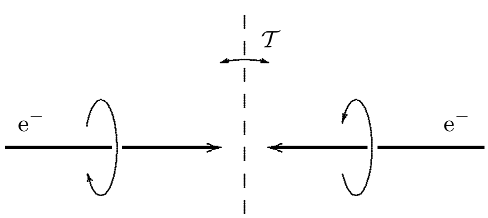

Fig. 5.2. $\mathcal{T}$ reverses the momentum of a particle and flips its spin

5.2.1 Time Inversion in Quantum Mechanics

In the Schrödinger representation, the state function satisfies the equation

$$\mathrm{i}\,\frac{\partial}{\partial t}\phi(t,\boldsymbol{x}) = H\phi(t,\boldsymbol{x})\,. \tag{5.39}$$

Invariance requires the transformed wave function $\mathcal{T}\phi(t,\boldsymbol{x})$ to satisfy the same equation with t replaced by $t' = -t$. The question is, how is $\mathcal{T}\phi(t,\boldsymbol{x})$ related to $\phi(t,\boldsymbol{x})$?

The simplest possible postulate, $\mathcal{T}\phi(t,\boldsymbol{x}) = \phi(-t,\boldsymbol{x})$, leads to

$$\mathrm{i}\,\frac{\partial}{\partial(-t)}\phi(-t,\boldsymbol{x}) = -H\phi(-t,\boldsymbol{x})\,, \tag{5.40}$$

so that, for example, the wave function of a free particle, $\phi(t,\boldsymbol{x}) = \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \mathrm{i}Et)$, will become after time inversion $\phi(-t,\boldsymbol{x}) = \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} + \mathrm{i}Et)$. For the dynamic equation to preserve its form, the Hamiltonian must be modified to $H' = -H$, which implies that to each state of positive energy before the transformation, there corresponds a state of negative energy after the transformation. States of negative energies are unstable and would sink to a state of infinitely large negative energy. The assumption $\mathcal{T}\phi(t,\boldsymbol{x}) = \phi(-t,\boldsymbol{x})$ is thus unacceptable for an invariant theory with positive energies before as well as after a time reversal.

The solution to this difficulty was given by Wigner in 1932. It is first assumed that there exists a *unitary* operator $\mathcal{U}$ such that $\mathcal{U}H^*\mathcal{U}^\dagger = H$ (where * means complex conjugation). Applying $\mathcal{U}$ from the left on both sides of the complex conjugate of (39) yields

$$\mathrm{i}\,\frac{\partial}{\partial(-t)}\mathcal{U}\phi^*(t,\boldsymbol{x}) = \mathcal{U}H^*\phi^*(t,\boldsymbol{x}) = H\mathcal{U}\phi^*(t,\boldsymbol{x})\,,$$

or, changing the sign of t,

$$\mathrm{i}\,\frac{\partial}{\partial t}\mathcal{U}\phi^*(-t,\boldsymbol{x}) = H\mathcal{U}\phi^*(-t,\boldsymbol{x})\,. \tag{5.41}$$

Thus, if $\phi(t,\boldsymbol{x})$ is a solution to (39), so too is the transformed wave function

$$\phi'(t',\boldsymbol{x}') = \mathcal{U}\phi^*(-t,\boldsymbol{x})\,. \tag{5.42}$$

In particular, if H is real, invariance to $\mathcal{T}$ means that if $\phi_E(\boldsymbol{x})$ represents a stationary wave function of energy E, the function $\phi_E^*(\boldsymbol{x})$ is also an energy eigenfunction with the same energy. This implies that if E is nondegenerate, then $\phi_E(\boldsymbol{x}) \propto \phi_E^*(\boldsymbol{x})$ and thus can be chosen real.

The time inversion operator on state vector space is thus the product of two operators: a unitary transformation $\mathcal{U}$, which replaces the state vector on which it operates with a time-reversed state vector, and a complex conjugation K of all the coefficients that may come with the state vector,

$$\mathcal{T}\,\alpha\psi(t) = \mathcal{U}K\,\alpha\psi(-t) = \alpha^*\,\mathcal{U}\psi^*(-t)\,. \tag{5.43}$$

The presence of a nontrivial $\mathcal{U}$ is necessary except in the most trivial cases. Since $K^2 = 1$ one has $\mathcal{T}^{-1} = K\mathcal{U}^\dagger$. Moreover, $\mathcal{T}$ has several properties worth noting:

(P1) *antilinearity*: $\mathcal{T}(a\,|\phi\rangle + b\,|\psi\rangle) = a^*\mathcal{T}\,|\phi\rangle + b^*\mathcal{T}\,|\psi\rangle$, where a and b are complex constants;

(P2) *antiunitarity*: $\langle \mathcal{T}\phi(t)\,|\mathcal{T}\psi(t)\,\rangle = \langle \phi(-t)\,|\psi(-t)\,\rangle^*$, a distinctive property of $\mathcal{T}$; but the norms of vectors and the probabilities remain invariant, just as in the case of unitary operators;

(P3) *operator transformation*: $\mathcal{O}' = \mathcal{T}\mathcal{O}\mathcal{T}^{-1} = \mathcal{U}\mathcal{O}^*\mathcal{U}^{-1}$ for an arbitrary operator $\mathcal{O}$, so that $\langle \mathcal{T}\psi(t)\,|\,\mathcal{O}'\,|\,\mathcal{T}\phi(t)\rangle = \langle \psi(-t)\,|\,\mathcal{O}\,|\,\phi(-t)\rangle^*$.

Example 1. Spin-0 Particle

For a spinless particle, $\mathcal{T}$ is merely complex conjugation K. Thus, its wave function in the x representation $\langle x\,|p\,\rangle = \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \mathrm{i}Et)$ becomes $\langle x'\,|p\,\rangle^* = \exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x} - \mathrm{i}Et)$. Also since $\langle Kx\,|p\rangle^* = \langle x\,|\,K\,|\,p\rangle$, the ket of a particle of momentum $\boldsymbol{p}$ transforms into

$$\mathcal{T}\,|\boldsymbol{p}\rangle = |-\boldsymbol{p}\rangle\ . \tag{5.44}$$

The radial part of a state having a well-defined angular momentum remains unchanged under $\mathcal{T}$, but its angular part $\langle x\,|\ell\, m\,\rangle = \mathrm{i}^\ell Y_{\ell m}(\theta,\varphi)$ transforms into

$$\langle x'\,|\ell\, m\,\rangle^* = (-\mathrm{i})^\ell\, Y^*_{\ell m}(\theta,\varphi) = (-)^{\ell-m}\,\mathrm{i}^\ell\, Y_{\ell,-m}(\theta,\varphi)\,,$$

which we may identify with $\langle x\,|\,\mathcal{T}\,|\,\ell\, m\rangle$ to get

$$\mathcal{T}\,|\ell\, m\rangle = (-)^{\ell-m}\,|\ell,-m\rangle\ . \tag{5.45}$$

This result agrees with the expected transformation of the angular momentum, $\boldsymbol{L}\to-\boldsymbol{L}$.

Example 2. Spin-1/2 Particle

By extension of the transformation rule for orbital angular momentum, it is assumed that the spin transforms as $\mathcal{T}\,\boldsymbol{S}\,\mathcal{T}^{-1} = -\boldsymbol{S}$, where $\mathcal{T} = \mathcal{U}K$. For spin $1/2$, $\boldsymbol{S} = \boldsymbol{\sigma}/2$. In the standard representation, where σ_x and σ_z are real and σ_y is imaginary, we have

$$\begin{aligned}
\mathcal{T}\,\sigma_x\,\mathcal{T}^{-1} &= \ \ \mathcal{U}\,\sigma_x\,\mathcal{U}^{-1} = -\sigma_x\,,\\
\mathcal{T}\,\sigma_y\,\mathcal{T}^{-1} &= -\mathcal{U}\,\sigma_y\,\mathcal{U}^{-1} = -\sigma_y\,,\\
\mathcal{T}\,\sigma_z\,\mathcal{T}^{-1} &= \ \ \mathcal{U}\,\sigma_z\,\mathcal{U}^{-1} = -\sigma_z\,.
\end{aligned}$$

These equations admit as solution $\mathcal{U} = \eta\sigma_y$, with η an arbitrary unimodular phase factor, $|\eta| = 1$. One may choose for example $\eta = -\mathrm{i}$ to reproduce the conventional phase of angular momentum eigenvectors. Thus, for the basis vectors

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

one gets $-\mathrm{i}\sigma_y \chi_m = (-)^{1/2-m}\chi_{-m}$, where $m = \pm 1/2$. Therefore, the state vector of a spin-$1/2$ particle of momentum $\boldsymbol{p}$ and polarization m transforms under $\mathcal{T}$ into

$$\mathcal{T}\ |\boldsymbol{p}, m\rangle = (-)^{1/2-m}\ |-\boldsymbol{p}, -m\rangle\ . \tag{5.46}$$

More generally, a vector of total angular momentum j, $j_z = m$ obeys the relation

$$\mathcal{T}\ |\alpha, j\, m\rangle = (-)^{j-m}\ |\alpha_{\mathrm{T}}, j, -m\rangle\ , \tag{5.47}$$

where α_{T} stands for the time-reversed quantum numbers corresponding to α. Note that $\mathcal{T}^2\ |j\, m\rangle = (-)^{2j}\ |j\, m\rangle$, and hence

$$\begin{aligned} \mathcal{T}^2 &= +1 \quad \text{for an integral spin particle, and} \\ \mathcal{T}^2 &= -1 \quad \text{for a half-integral spin particle .} \end{aligned}$$

This is a general result which depends neither on the phase convention nor on the representation of state vectors. In particular, for a system of N fermions, $\mathcal{T}^2 = (-)^N$. ■

5.2.2 Time Inversion in Field Theories

We begin by studying the behavior of classical c-numbered fields under time inversion and generalize the results to the corresponding field operators.

Scalar Fields. The Klein–Gordon equation for a classical scalar field ϕ_{c} is invariant to time inversion whether the transformation rule is $\phi'_{\mathrm{c}}(t') = \phi_{\mathrm{c}}(-t)$ or $\phi'_{\mathrm{c}}(t') = \phi^*_{\mathrm{c}}(-t)$. However, for consistency we adopt the same rule as for the Schrödinger equation, up to an arbitrary phase,

$$\mathcal{T}\ : \phi_{\mathrm{c}}(t, \boldsymbol{x}) \to \phi'_{\mathrm{c}}(t', \boldsymbol{x}') = \zeta_{\mathrm{B}}\phi^*_{\mathrm{c}}(-t, \boldsymbol{x}), \qquad \text{such that } |\zeta_{\mathrm{B}}| = 1\ . \tag{5.48}$$

As seen in Chap. 2, this c-numbered function is related to the corresponding quantized field $\phi(x)$ by

$$\phi_{\mathrm{c}}(t, \boldsymbol{x}) = \langle 0\,|\,\phi(t, \boldsymbol{x})\,|\,\boldsymbol{p}\rangle\ , \tag{5.49}$$

which, upon application of $\mathcal{T}$ on both sides, results in

$$\zeta_{\mathrm{B}}\phi^*_{\mathrm{c}}(-t, \boldsymbol{x}) = \langle \mathcal{T}0\,|\,\phi(t, \boldsymbol{x})\,|\,\mathcal{T}\boldsymbol{p}\rangle\ . \tag{5.50}$$

By property P3, the left-hand side is $\zeta_{\mathrm{B}}\left\langle\mathcal{T}0\,\middle|\,\mathcal{T}\,\phi(-t,\boldsymbol{x})\,\mathcal{T}^{-1}\,\middle|\,\mathcal{T}\boldsymbol{p}\right\rangle$, which leads to

$$\mathcal{T}\,\phi(t,\boldsymbol{x})\,\mathcal{T}^{-1}=\zeta_{\mathrm{B}}\phi(-t,\boldsymbol{x})\,,\qquad(\zeta_{\mathrm{B}}=\pm1).\tag{5.51}$$

Note the transformed quantized field is *not* complex conjugated.

To obtain the transformation rules for Fock operators we substitute the plane-wave expansion of ϕ (2.99) into (51) to obtain for its left-hand side

$$\mathcal{T}\,\phi(x)\,\mathcal{T}^{-1}=\sum_{p}C_{\boldsymbol{p}}\left[\mathcal{T}\,a_{\boldsymbol{p}}\,\mathcal{T}^{-1}\mathrm{e}^{\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\;+\;\mathcal{T}\,b^{\dagger}_{\boldsymbol{p}}\,\mathcal{T}^{-1}\mathrm{e}^{-\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\right]\tag{5.52}$$

(the exponentials having been complex conjugated by antilinearity), and for its right-hand side

$$\begin{aligned}\zeta_{\mathrm{B}}\,\phi(-t,\boldsymbol{x})&=\zeta_{\mathrm{B}}\sum_{p}C_{\boldsymbol{p}}\left[a_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}(Et+\boldsymbol{p}\cdot\boldsymbol{x})}\;+\;b^{\dagger}_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}(Et+\boldsymbol{p}\cdot\boldsymbol{x})}\right]\\&=\zeta_{\mathrm{B}}\sum_{p}C_{\boldsymbol{p}}\left[a_{-\boldsymbol{p}}\mathrm{e}^{\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\;+\;b^{\dagger}_{-\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}(Et-\boldsymbol{p}\cdot\boldsymbol{x})}\right]\end{aligned}\tag{5.53}$$

(changing $t\to-t$ and $\boldsymbol{p}\to-\boldsymbol{p}$ in the sum). Identifying the right-hand sides of the two resulting equations, one gets (with $\zeta_{\mathrm{B}}=\pm1$)

$$\begin{aligned}&\mathcal{T}\,a_{\boldsymbol{p}}\,\mathcal{T}^{-1}=\zeta_{\mathrm{B}}a_{-\boldsymbol{p}}\,,\\&\mathcal{T}\,b^{\dagger}_{\boldsymbol{p}}\,\mathcal{T}^{-1}=\zeta_{\mathrm{B}}b^{\dagger}_{-\boldsymbol{p}}\,.\end{aligned}\tag{5.54}$$

To illustrate, consider the probability amplitude for the presence of a boson of momentum $\boldsymbol{p}$ in an arbitrary state ψ,

$$\begin{aligned}\langle\psi\,|\boldsymbol{p}\rangle&=\left\langle\psi\,\middle|\,a^{\dagger}_{\boldsymbol{p}}\,\middle|\,0\right\rangle=\left\langle\mathcal{T}\psi\,\middle|\,\mathcal{T}\,a^{\dagger}_{\boldsymbol{p}}\,\mathcal{T}^{-1}\,\middle|\,\mathcal{T}0\right\rangle^{*}\\&=\zeta_{\mathrm{B}}\,\langle0\,|\,a_{-\boldsymbol{p}}\,|\,\mathcal{T}\psi\rangle\;=\zeta_{\mathrm{B}}\,\langle-\boldsymbol{p}\,|\mathcal{T}\psi\,\rangle\;.\end{aligned}$$

In the time-reversed amplitude the particle reverses its direction of motion, with the initial state found in the bra.

The current density for a scalar field (13) transforms into

$$\mathcal{T}\,j^{0}(t,\boldsymbol{x})\,\mathcal{T}^{-1}=+j^{0}(-t,\boldsymbol{x}),\qquad\mathcal{T}\,j^{i}(t,\boldsymbol{x})\,\mathcal{T}^{-1}=-j^{i}(-t,\boldsymbol{x}),\tag{5.55}$$

just as anticipated from classical arguments.

Electromagnetic Field. If the electromagnetic interaction is T-invariant as shown by observations, and if it may be described by $H_{\mathrm{em}}=qj^{\mu}A_{\mu}$, then given the property (55) of the current, $A^{\mu}(x)$ must satisfy

$$\mathcal{T}\,A^{\mu}(t,\boldsymbol{x})\,\mathcal{T}^{-1}=(A^{0}(-t,\boldsymbol{x}),\;-A^{i}(-t,\boldsymbol{x}))\,.\tag{5.56}$$

The two sides of this equation may be explicitly written out, making use of the plane-wave expansion series (2.156) for the transverse field $\boldsymbol{A}$,

$$\begin{aligned}\mathcal{T}\,\boldsymbol{A}(t,\boldsymbol{x})\,\mathcal{T}^{-1} = \sum_{k\lambda} C_{\boldsymbol{k}}\Big[&\boldsymbol{\epsilon}^*(\boldsymbol{k},\lambda)\,\mathcal{T}\,a(\boldsymbol{k},\lambda)\,\mathcal{T}^{-1}\mathrm{e}^{\mathrm{i}k\cdot x}\\ &+\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\,\mathcal{T}\,a^\dagger(\boldsymbol{k},\lambda)\,\mathcal{T}^{-1}\mathrm{e}^{-\mathrm{i}k\cdot x}\Big]\,;\end{aligned}\tag{5.57}$$

$$\begin{aligned}&-\boldsymbol{A}(-t,\boldsymbol{x})\\ &=-\sum_{k\lambda} C_{\boldsymbol{k}}\,\left[\boldsymbol{\epsilon}(-\boldsymbol{k},\lambda)\,a(-\boldsymbol{k},\lambda)\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}+\boldsymbol{\epsilon}^*(-\boldsymbol{k},\lambda)\,a^\dagger(-\boldsymbol{k},\lambda)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\right]\\ &=+\sum_{k\lambda} C_{\boldsymbol{k}}\,\left[\boldsymbol{\epsilon}^*(\boldsymbol{k},\lambda)\,a(-\boldsymbol{k},\lambda)\,\mathrm{e}^{\mathrm{i}k\cdot x}+\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\,a^\dagger(-\boldsymbol{k},\lambda)\,\mathrm{e}^{-\mathrm{i}k\cdot x}\,\right]\,.\end{aligned}\tag{5.58}$$

In the last step we have used (18). Identifying the right-hand sides of the two resulting equations leads to the transformation rules for the photon Fock operators:

$$\begin{aligned}\mathcal{T}\,a(\boldsymbol{p},\lambda)\,\mathcal{T}^{-1} &= a(-\boldsymbol{p},\lambda)\,,\\ \mathcal{T}\,a^\dagger(\boldsymbol{p},\lambda)\,\mathcal{T}^{-1} &= a^\dagger(-\boldsymbol{p},\lambda)\,.\end{aligned}\tag{5.59}$$

The result is consistent with a phase equal to +1. The photon helicity is unchanged by $\mathcal{T}$ because both its spin and momentum change signs.

Dirac Field. It is assumed, just as before, that the c-numbered Dirac wave function transforms under time inversion as

$$\mathcal{T}\;:\psi_{\mathrm{c}}(t,\boldsymbol{x})\to\psi'_{\mathrm{c}}(t',\boldsymbol{x}')=\zeta_{\mathrm{F}}\,A\psi^*_{\mathrm{c}}(-t,\boldsymbol{x}),\qquad |\zeta_{\mathrm{F}}|=1.\tag{5.60}$$

Here A is a 4×4 unitary matrix in terms of which any component of the transformed spinor is expressed as a linear combination of different components of the original spinor. However, in quantum field theory, if the rule $\psi\to A\psi^\dagger$ were adopted, a fermion would transform into an antifermion, which would not be physically acceptable. To find the correct transformation rules, one follows the same arguments as for the boson field and obtains

$$\begin{aligned}\mathcal{T}\,\psi(t,\boldsymbol{x})\,\mathcal{T}^{-1} &= \zeta_{\mathrm{F}}\,A\,\psi(-t,\boldsymbol{x})\,,\\ \mathcal{T}\,\psi^\dagger(t,\boldsymbol{x})\,\mathcal{T}^{-1} &= \zeta_{\mathrm{F}}^*\,\psi^\dagger(-t,\boldsymbol{x})A^\dagger\,,\\ \mathcal{T}\,\overline{\psi}(t,\boldsymbol{x})\,\mathcal{T}^{-1} &= \zeta_{\mathrm{F}}^*\,\overline{\psi}(-t,\boldsymbol{x})A^\dagger\,.\end{aligned}\tag{5.61}$$

The matrix A cannot be calculated by the same relation which has served to determine $S(a)$ for parity, because time inversion is antiunitary whereas parity is unitary. Rather, one proceeds by imposing T-invariance on the Dirac Lagrangian,

$$\mathcal{T}\,\mathcal{L}_{\mathrm{F}}(x)\,\mathcal{T}^{-1}=\mathcal{L}_{\mathrm{F}}(x')\,,\qquad \text{where }\;x'=(-t,\boldsymbol{x})\,.\tag{5.62}$$

From (61) and properties P1–P3 of $\mathcal{T}$, one gets for the right-hand side

$$\mathcal{L}_{\mathrm{F}}(x') = \overline{\psi}(x')(\mathrm{i}\gamma^\mu \partial'_\mu - m)\psi(x')\,,$$

and for the left-hand side

$$\begin{aligned}\mathcal{T}\,\mathcal{L}_{\mathrm{F}}(x)\,\mathcal{T}^{-1} &= \mathcal{T}\,\psi^\dagger(x)\,\mathcal{T}^{-1}\left[-\mathrm{i}(\gamma^0\gamma^\mu)^*\partial_\mu - m\gamma^{0*}\right]\mathcal{T}\,\psi(x)\,\mathcal{T}^{-1} \\ &= \overline{\psi}(x')\gamma^0 A^\dagger\left(\mathrm{i}\gamma^{0\mathrm{T}}\gamma^{\mu\mathrm{T}}\partial'_\mu - m\gamma^{0\mathrm{T}}\right)A\psi(x')\,,\end{aligned}$$

where use has been made of the matrix properties $\gamma^{\mu\mathrm{T}} = (\gamma^0\gamma^\mu\gamma^0)^*$, or $\gamma^{0\mathrm{T}} = \gamma^{0*}$ and $\gamma^{i\mathrm{T}} = -\gamma^{i*}$, which follow from their Hermitian conjugation property, $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. From these relations one can immediately infer the defining property of A :

$$A\gamma^\mu A^\dagger = \gamma^{\mu\mathrm{T}} \qquad \text{(in arbitrary representation).} \tag{5.63}$$

To find an explicit expression for A, a concrete representation of γ_μ is needed. In the standard representation where only γ^2 is imaginary, A commutes with γ^0 and γ^2, and anticommutes with both γ^1 and γ^3 :

$$\begin{aligned}&A\gamma^0 = \gamma^0 A, \quad A\gamma^1 = -\gamma^1 A\,,\\ &A\gamma^2 = \gamma^2 A, \quad A\gamma^3 = -\gamma^3 A\,.\end{aligned}$$

These conditions hold provided that

$$A = \lambda\gamma^1\gamma^3, \quad |\lambda|^2 = 1 \qquad \text{(in standard representation)}\ . \tag{5.64}$$

As an application of this result, consider the fermion current $j_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$, which transforms into

$$\begin{aligned}\mathcal{T}\,j_\mu(x)\,\mathcal{T}^{-1} &= |\zeta_{\mathrm{F}}|^2\,\psi^\dagger(x')A^\dagger(\gamma_0\gamma_\mu)^*A\psi(x') \\ &= \overline{\psi}(x')A^\dagger\gamma_\mu^* A\psi(x') = \overline{\psi}(x')\gamma_\mu^\dagger\psi(x')\,.\end{aligned}$$

This reduces to (55) on using (63) for A. Transformations of other bilinear covariants, which may describe various interaction models, can be similarly found. Thus, for example,

$$\begin{aligned}\mathcal{T}\,\overline{\psi}(x)\psi(x)\,\mathcal{T}^{-1} &= \overline{\psi}(x')A^\dagger A\psi(x') \;=\; \overline{\psi}(x')\psi(x')\,;\\ \mathcal{T}\,\overline{\psi}(x)\gamma^\mu\psi(x)\,\mathcal{T}^{-1} &= \overline{\psi}(x')A^\dagger(\gamma^\mu)^*A\psi(x') \;=\overline{\psi}(x')\gamma^{\mu\dagger}\psi(x')\,;\\ \mathcal{T}\,\overline{\psi}(x)\mathrm{i}\gamma_5\psi(x)\,\mathcal{T}^{-1} &= \overline{\psi}(x')A^\dagger(\mathrm{i}\gamma_5)^*A\psi(x') = -\overline{\psi}(x')\mathrm{i}\gamma_5\psi(x')\,;\\ \mathcal{T}\,\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)\,\mathcal{T}^{-1} &= \overline{\psi}(x')A^\dagger\gamma^{\mu*}\gamma_5^*A\psi(x') = \overline{\psi}(x')\gamma^{\mu\dagger}\gamma_5\psi(x')\,.\end{aligned}$$

Let us now examine the action of A on spinors of polarizations $s = \pm 1/2$. With the phase conventions adopted in (3.45) and (3.46), we have

$$\begin{aligned}&Au(\boldsymbol{p},s) = -\lambda\,(-)^{1/2-s}u^*(-\boldsymbol{p},-s)\,,\\ &Av(\boldsymbol{p},s) = -\lambda\,(-)^{1/2-s}v^*(-\boldsymbol{p},-s)\,.\end{aligned} \tag{5.65}$$

In order to determine the transformations of the Fock operators for fermions, we write out the expansion series of (61). We have, on the one hand,

$$\begin{aligned}\mathcal{T}\,\psi(x)\,\mathcal{T}^{-1} =& \sum_{\boldsymbol{p},s} C_{\boldsymbol{p}}\big[\mathcal{T}\,b(\boldsymbol{p},s)\,\mathcal{T}^{-1}u^*(\boldsymbol{p},s)\,\mathrm{e}^{\mathrm{i}p\cdot x} \\ &+ \mathcal{T}\,d^\dagger(\boldsymbol{p},s)\,\mathcal{T}^{-1}v^*(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\big]\end{aligned} \tag{5.66}$$

(using antilinearity of $\mathcal{T}$), and on the other hand,

$$\begin{aligned}A\psi(-t,\boldsymbol{x}) =& \lambda\sum_{\boldsymbol{p},s}(-)^{1/2-s}C_{\boldsymbol{p}}[b(-\boldsymbol{p},-s)u^*(\boldsymbol{p},s)\mathrm{e}^{\mathrm{i}p\cdot x} \\ &+ d^\dagger(-\boldsymbol{p},-s)v^*(\boldsymbol{p},s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,]\,,\end{aligned} \tag{5.67}$$

where we have used (65) and changed the signs of $\boldsymbol{p}$ and s in the sums; since s is half-integral, $(-)^{1/2+s} = -(-)^{1/2-s}$. From these results follow the relations

$$\mathcal{T}\,b(\boldsymbol{p},s)\,\mathcal{T}^{-1} = (-)^{1/2-s}\zeta_{\mathrm{F}}\,b(-\boldsymbol{p},-s)\,, \tag{5.68}$$

$$\mathcal{T}\,d^\dagger(\boldsymbol{p},s)\,\mathcal{T}^{-1} = (-)^{1/2-s}\zeta_{\mathrm{F}}\,d^\dagger(-\boldsymbol{p},-s)\,, \tag{5.69}$$

where $\lambda \equiv +1$. This choice is adopted simply to be in accord with (46),

$$\mathcal{T}\;|\boldsymbol{p},s\rangle = (-)^{1/2-s}\;|-\boldsymbol{p},-s\rangle\;. \tag{5.70}$$

5.2.3 $\mathcal{T}$ and Interactions

The interaction Hamiltonian $H_{\mathrm{em}} = qj_\mu A^\mu$ describes electromagnetic phenomena. As we have seen, it is invariant to time inversion. An example of noninvariant interaction is that of the electric dipole moment which is described in the nonrelativistic limit by $H_{\mathrm{d}} = -\mu^{(\mathrm{el})}\boldsymbol{\sigma}\cdot\boldsymbol{E}$. Since, under time inversion, the electric field remains unchanged while the angular momentum reverses its direction, H_{d} is odd with respect to $\mathcal{T}$ (as it is also with respect to $\mathcal{P}$). Hermiticity of the Hamiltonian requires $\mu^{(\mathrm{el})}$ to be real. Experiments have proved that this interaction is negligible, as shown by the extremely small values of the measured electric dipole moments of typical fermions:

e	$(-0.3 \pm 0.8) \times 10^{-26}\;e\,\mathrm{cm}\,,$
μ	$(3.7 \pm 3.4) \times 10^{-19}\;e\,\mathrm{cm}\,,$
p	$(-4 \pm 6) \times 10^{-23}\;e\,\mathrm{cm}\,,$
n	$< 1.1 \times 10^{-25}\;e\,\mathrm{cm}\;.$

Just as with other symmetries, T-invariance imposes restrictions on physical models. Consider for example a possible candidate for the interaction model of fermionic fields with a neutral scalar or pseudoscalar field. Nonderivative Yukawa couplings may take the general form

$$\mathcal{H}_{\mathrm{int}} = (g_{\mathrm{s}}\overline{\psi}\psi + \mathrm{i}g_{\mathrm{p}}\overline{\psi}\gamma_5\psi)\phi\,. \tag{5.71}$$

As $\mathcal{H}_{\rm int}$ is Hermitian, the coupling constants $g_{\rm s}$ and $g_{\rm p}$ must be real. The time-reversed Hamiltonian is

$$\mathcal{T}\,\mathcal{H}_{\rm int}\,\mathcal{T}^{-1} = (g_{\rm s}\overline{\psi}\psi - {\rm i}g_{\rm p}\overline{\psi}\gamma_5\psi)\zeta_\phi\phi\,. \tag{5.72}$$

To have a T-invariant model, clearly one of the two terms must be absent. In this example, invariance of the model to time inversion also implies its invariance to space inversion.

An arbitrary quantum state is normally a very complex quantity, and only the state vectors of a stable particle are simple enough for the time-reversed vectors to be explicitly known. In general, the time reverse of an arbitrary state is complicated and its actual observation, unlikely. For this reason a direct verification of T-invariance of a dynamic equation is difficult. More often, the symmetry can only be indirectly tested, for example by the confirmation or invalidation of certain predicted phase relations. Let us consider again the β-decay Hamiltonian (37), where both $C_{\rm V}$ and $C_{\rm A}$ are now assumed to be *a priori* complex. Under time inversion,

$$\begin{aligned}&\mathcal{T}\left[\overline{\psi}_{\rm p}\gamma_\mu(C_{\rm V} - C_{\rm A}\gamma_5)\psi_{\rm n}\right]\left[\overline{\psi}_{\rm e}\gamma^\mu(1-\gamma_5)\psi_\nu\right]\mathcal{T}^{-1}\\&\quad = \zeta_{\rm p}^*\zeta_{\rm e}^*\zeta_{\rm n}\zeta_\nu\left[\overline{\psi}_{\rm p}\gamma_\mu(C_{\rm V}^* - C_{\rm A}^*\gamma_5)\psi_{\rm n}\right]\left[\overline{\psi}_{\rm e}\gamma^\mu(1-\gamma_5)\psi_\nu\right].\end{aligned} \tag{5.73}$$

So the interaction (37) is T-invariant provided the constants $C_{\rm A}$ and $C_{\rm V}$ are relatively real. This condition can be experimentally checked. Available data

$$C_{\rm A}/C_{\rm V} \equiv |C_{\rm A}/C_{\rm V}|\exp({\rm i}\phi_{\rm AV}), \quad \phi_{\rm AV}^{\rm exp} = (180.07 \pm 0.18)^\circ\,, \tag{5.74}$$

demonstrate without ambiguities that the β-decay is time-reversal-invariant. However, not all weak interactions have this property. In particular, a small but unmistakable violation of the T-symmetry has been detected in the K^0–$\bar{\mathrm{K}}^0$ system (see Chap. 11).

5.3 Charge Conjugation

We have considered up to now symmetries in space-time and their associated quantum numbers. In addition to these numbers of kinematic origins, particles may have other attributes, called *internal*, such as the electric charge, the baryon number, the lepton number, and the strangeness. These numbers may also obey conservation rules that arise from the fact that the phases of non-Hermitian fields are nonobservable or, equivalently, from the invariance of the theory to such phase or gauge transformations. Each internal quantum number is associated with an abstract internal space in which symmetry operations, such as rotations or reflections, can be defined in analogy with similar operations in ordinary space. In this section, we will first introduce some such internal quantum numbers (also called generalized charges) and then discuss the *charge conjugation*, which is a discrete transformation that reverses the signs of all nonzero generalized charges of a given particle, converting it into the corresponding antiparticle.

5.3.1 Additive Quantum Numbers

We have seen in Chap. 2 that invariance of a theory to a global gauge transformation independent of space-time coordinates gives rise to a local conserved current and an associated charge, which is constant in time. Such a transformation effects a phase change on the wave function of the first-quantized formalism,

$$\varphi \to \exp(-\mathrm{i}q\alpha)\varphi\,,$$

or on the field operator in the second-quantized formalism,

$$U\varphi U^{-1} = \mathrm{e}^{-\mathrm{i}q\alpha}\varphi\,, \quad U = \mathrm{e}^{\mathrm{i}Q\alpha}\,. \tag{5.75}$$

Here α is a real arbitrary constant and q the electric charge of the field. The operator Q which generates infinitesimal gauge transformations is identified with the electric charge operator and is, for this reason, Hermitian. Invariance of the theory means that the system remains unchanged on application of U and that the total charge is conserved. This may be expressed for example in terms of the invariance of the S-matrix with respect to arbitrary phase transformations of the charged states, so that a typical S-matrix element for an allowed transition varies as

$$\langle \mathrm{f}\,|\,S\,|\,\mathrm{i}\rangle \to [1 + \mathrm{i}(Q_\mathrm{i} - Q_\mathrm{f})\alpha\,]\,\langle \mathrm{f}\,|\,S\,|\,\mathrm{i}\rangle\;,$$

where Q_i (Q_f) is the total charge of the initial (final) state. Invariance implies that S is unchanged, and hence that $Q_\mathrm{i} = Q_\mathrm{f}$. If the system is composed of particles of charges q_1, $q_2, \ldots$, then $U \to \exp[-\mathrm{i}\alpha(q_1 + q_2 + \ldots)]$ when applied on the state vector of the system, and the total charge is the algebraic sum of all individual charges $(q_1 + q_2 + \ldots)$. To the Q operator corresponds then an *additive* quantum number, as is the general case of any Hermitian operator that generates a unitary transformation by exponentiation.

The evidence for the *electric charge conservation* is strong. If this conservation rule were broken, the lightest charged particle, the electron, would decay into lighter neutral particles in processes such as $\mathrm{e} \to \nu\gamma$ or $\mathrm{e} \to \nu\nu\bar{\nu}$. Data show that these processes rarely occur, if at all. The current value of the mean lifetime of the electron is $\tau_\mathrm{e} > 2.7 \times 10^{23}$ years (to be compared with the estimated age of the universe, $t_0 \approx 10^{10}$ years). In other words, the electron appears to be stable.

A remarkable property of the electric charge is its quantization: every *observable* particle carries an electric charge which is a whole multiple of the unit charge, $|q| = Ne$, where $N = 0, 1, 2, \ldots$. In particular, the neutron charge must be exactly zero and the charges of the electron and proton must be equal in magnitudes and opposite in signs. Data confirm this expectation:

$$q_\mathrm{n} = (-0.4 \pm 1.1) \times 10^{-21}\,e\,,$$
$$|q_\mathrm{p} + q_\mathrm{e}| < 1.0 \times 10^{-21}\,e\,.$$

As we have seen before, a free neutron decays into a proton, an electron, and an antineutrino. In this process as well as in any other reactions involving nucleons, the number of nucleons is always conserved regardless of the interactions involved. Although free neutrons decay, free protons are believed to be stable. This general property of matter is encoded in a new additive conserved quantum number, called the *baryon number* N_{B}, which takes value $N_{\mathrm{B}} = +1$ for n, p, Λ^0, $\Sigma^{\pm,0}$, and Ξ^{-0}, and $N_{\mathrm{B}} = -1$ for the corresponding antiparticles. Quarks and antiquarks are assigned fractional baryon numbers. Finally, the photon, mesons, and leptons all have vanishing baryon numbers.

Conservation of the baryon number implies that an antibaryon cannot be produced alone from baryons, but always in association with another baryon. For example,

$$\begin{aligned} \mathrm{p} + \mathrm{p} &\to \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}} \,, \\ \pi^- + \mathrm{p} &\to \mathrm{p} + \bar{\mathrm{p}} + \Lambda^0 + \mathrm{K}^0 \,. \end{aligned}$$

Just as for the electric charge, conservation of the baryon number is practically absolute, which implies that no matrix elements of a physical operator can exist between states of different baryon numbers. As far as we know, matter is stable and, in particular, the lightest baryon, the proton, is believed to be stable. Estimates of the mean lifetime of the proton vary with the assumed decay modes,

$$\begin{aligned} \tau_{\mathrm{p}} &> 1.6 \times 10^{25}\,\text{years} \qquad \text{(mode independent)} \,, \\ &> 10^{31} - 5 \times 10^{32}\,\text{years} \quad \text{(mode dependent)} \,. \end{aligned}$$

But, regardless of the decay modes considered, these limits are always by far greater than the age of the universe.

The observability of processes involving leptons can be similarly encoded in various *lepton numbers* L_ℓ, L_{e}, L_μ, and L_τ, whose values (+1 for each type of lepton and −1 for each antilepton) are assigned to different particles according to Table 5.1 .

Conservation of the lepton numbers implies that leptons are created or destroyed in charge-conjugate pairs. Thus, in the neutron β-decay process $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$, an electronic antineutrino appears together with an electron while in the inverse β-decay of a bound proton, $\mathrm{p} \to \mathrm{n} + \mathrm{e}^+ + \nu_{\mathrm{e}}$ (for example in the nuclear transition $^{14}\mathrm{O} \to {}^{14}\mathrm{N}^* + \mathrm{e}^+ + \nu_{\mathrm{e}}$), an electronic neutrino and a positron are emitted. It is also possible, when enough energy is available, to observe an L_{e}-conserving double-β-decay in which two (bound) neutrons decay,

$$\begin{array}{cccccc} & 2\mathrm{n} & \to & 2\mathrm{p} & +2\mathrm{e}^- & +2\bar{\nu}_{\mathrm{e}} \,, \\ L_{\mathrm{e}} : & 0 & & 0 & +2 & -2 \end{array}$$

(as in the nuclear transition $^{48}\mathrm{Ca} \to {}^{48}\mathrm{Ti} + 2\mathrm{e}^- + 2\bar{\nu}_{\mathrm{e}}$), giving rise to two antineutrinos in the final state. These elusive particles would be completely

Table 5.1. Lepton numbers [a]

	e^-, ν_e	$e^+, \bar{\nu}_e$	μ^-, ν_μ	$\mu^+, \bar{\nu}_\mu$	τ^-, ν_τ	$\tau^+, \bar{\nu}_\tau$
L_e	+1	−1	0	0	0	0
L_μ	0	0	+1	−1	0	0
L_τ	0	0	0	0	+1	−1
L_ℓ	+1	−1	+1	−1	+1	−1

[a] All other particles have zero lepton numbers.

absent if the lepton number L_e were not conserved, because then an L_e-violating decay could first take place via $n \to p + e^- + \nu_e$ (in which ν_e rather than $\bar{\nu}_e$ is produced), to be followed by an L_e-conserving reaction, $\nu_e + n \to p + e^-$, which gives the net result

$$\begin{array}{lcccc} & 2n & \to & 2p & +2e^- \,. \\ L_e : & 0 & & 0 & +2 \end{array}$$

Experiments show that neutrinoless double-β-decays are by far less probable than the corresponding neutrino-emitting decays of the same nuclei.

Historically, the hypothesis of the existence of an additional lepton number, $L_\mu \neq L_e$, was introduced to explain the suppression of the decay mode

$$\begin{array}{lcccc} & \mu^+ & \to & e^+ & \gamma\,, \\ L_e : & 0 & & -1 & 0 \\ L_\mu : & -1 & & 0 & 0 \\ L_\ell : & -1 & & -1 & 0 \end{array}$$

a process which would otherwise not be forbidden by the conservation rules of the generic lepton number L_ℓ and of any other known quantum number. The experimentally observed suppression of the process,

$$\frac{\tau(\mu \to e\,\gamma)}{\tau(\mu \to e\,\bar{\nu}\,\nu)} < 2 \times 10^{-8}\,, \tag{5.76}$$

implies that if a reaction is initiated by a μ-neutrino, it must produce a muon as in $\nu_\mu + n \to \mu^- + p$, rather than an electron, as it would be the case in $\nu_\mu + n \to e^- + p$. This hypothesis is strongly supported by the much higher probability of observing the decay mode $\pi^+ \to \mu^+ \nu_\mu$ (99.98% of all modes) compared to $\pi^+ \to \mu^+ \nu_e$ (8.0×10^{-3} of all modes). The existence of the muonic neutrino distinct from the electronic neutrino is thus confirmed and motivates the introduction of a new additive quantum number $L_\mu \neq L_e$.

The τ lepton was discovered in 1975 in the reaction $e^+ + e^- \to \tau^+ + \tau^-$, followed by the decays $\tau^- \to \ell^- + \bar{\nu}_\ell + \nu_\tau$ and $\tau^+ \to \ell^+ + \nu_\ell + \bar{\nu}_\tau$, where $\ell = e, \mu$. As experiments show that the process $\nu_\mu + n \to \tau^- + p$ is highly improbable, it is concluded that $\nu_\mu \neq \nu_\tau$ and, by the same token, also $\nu_e \neq \nu_\tau$. If that is the case, there must exist a τ-lepton number L_τ.

While all quantum numbers discussed in this section are additively conserved, the electric charge is outstanding in its remarkable particularity of playing the double role of being an additive quantum number by its presence in the *global gauge* transformation $U = \exp(iQ\alpha)$, and of being a coupling constant of an interaction by its presence in the *local gauge* transformation $U(x) = \exp[iQ\alpha(x)]$ (as we shall see in detail in Chap. 8). In contrast, neither the baryon number nor the lepton numbers seem to be associated with detectable interactions. It follows that the electric charge alone can be expressed in terms of a measurable physical unit, while the baryon number and the lepton numbers have arbitrary units. The profound implication of this difference is that the conservation of electric charge is exact, being anchored by a principle considered as fundamental – the local gauge invariance – whereas the conservations of the baryon and lepton numbers may be approximate. Thus, in principle, one may not exclude, for example, processes like $p \to e^+\pi^0$, $p \to e^+\gamma$, or $\nu_e \leftrightarrow \nu_\mu$, $\nu_\mu \leftrightarrow \nu_\tau$.

Finally, leaving aside for the moment quantum numbers of more recent origins (charm, topness, bottomness), we now consider briefly another additive quantum number called *strangeness*. It is a quantum number assigned to some hadrons (baryons or mesons) that have apparently contradictory properties. These particles are copiously produced in nucleon–nucleus collisions and other hadronic reactions with total production cross-sections of the order of a millibarn, comparable in magnitude to cross-sections for other strong interaction processes, such as pion–nucleon reactions. However, their rather long lifetimes, typically $\tau \approx 10^{-10}\,$s, suggest that their decays arise from weak interactions.

The hypothesis of strangeness S gives a simple solution to this dilemma by assuming that S is conserved in strong interactions, responsible for productions, but is nonconserved in weak interactions, responsible for decays. Strangeness is thus associated with an imperfect symmetry, in contrast to the other additive quantum numbers studied in this section. Given this basic assumption, the values of S for all hadrons can be determined relatively to a few selected reference particles:

$$\begin{aligned} S &= \ 0 \qquad \text{for } \pi, \text{ nucleons}\ ; \\ S &= +1 \qquad \text{for } \mathrm{K}^+\ . \end{aligned}$$

Mesons K^+, among the first strange particles observed, are produced in

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{K}^+ + \Lambda^0\,.$$

Assuming conservation of strangeness in this production reaction, one obtains $S = -1$ for Λ^0. The quantum number S for other hadrons is determined in a similar fashion:

$$\begin{aligned}
&\pi^-\,\mathrm{p} \to \mathrm{K}^0\,\Lambda^0 && S(\mathrm{K}^0) = +1\,,\\
&\pi^-\,\mathrm{p} \to \mathrm{n}\,\mathrm{K}^+\,\mathrm{K}^- && S(\mathrm{K}^-) = -1\,,\\
&\pi^-\,\mathrm{p} \to \mathrm{K}^+\Sigma^- && S(\Sigma^-) = -1\,,\\
&\mathrm{p}\,\mathrm{p} \quad \to \mathrm{n}\,\mathrm{K}^+\Sigma^+ && S(\Sigma^+) = -1\,,\\
&\mathrm{pp} \quad \to \mathrm{p}\,\mathrm{K}^+\Sigma^0 && S(\Sigma^0) = -1\,,\\
&\pi^-\mathrm{p} \ \to \mathrm{K}^+\mathrm{K}^0\,\Xi^- && S(\Xi^-) = -2\,,\\
&\pi^+\mathrm{p} \ \to \mathrm{K}^+\mathrm{K}^+\Xi^0 && S(\Xi^0) = -2\,.
\end{aligned}$$

Note that the baryons Σ^+, Σ^- and Σ^0 have the same value $S = -1$ as well as the same mass (see Table 1.3). This fact points to the existence of some new symmetry (which will be identified as the isospin symmetry in the following chapter). As for the mesons K, the situation is somewhat different. Mesons K^+ and K^- have equal masses but electric charges and strangeness numbers of opposite signs, which indicates that they are charge conjugates to each other. It is then plausible that K^0 must similarly have a charge conjugate of strangeness $S = -1$ and equal mass. It is possible to detect such a particle, called $\bar{\mathrm{K}}^0$, for example in

$$\pi^+\mathrm{p} \to \mathrm{p}\,\bar{\mathrm{K}}^0\,\mathrm{K}^+\,.$$

Observations indicate that there must exist two doublets of mesons of $S = \pm 1$ conjugate to each other, $(\mathrm{K}^+, \mathrm{K}^0)$ and $(\mathrm{K}^-, \bar{\mathrm{K}}^0)$. Similarly, the doubly strange baryons Ξ^- and Ξ^0 form a doublet to which corresponds a distinct antidoublet. Finally, there exists a particle, Ω^- (1672 MeV), with $S = -3$.

Electromagnetically induced reactions, such as

$$\begin{aligned}
&\gamma\,\mathrm{p} \to \Sigma^0\,\mathrm{K}^+, \quad (S_\mathrm{i} = S_\mathrm{f} = 0)\,,\\
&\gamma\,\mathrm{p} \to \Sigma^+\mathrm{K}^0, \quad (S_\mathrm{i} = S_\mathrm{f} = 0)\,,
\end{aligned}$$

have been observed but not, for example,

$$\begin{aligned}
&\gamma\,\mathrm{p} \to \Lambda^0\,\pi^+, \quad\ (S_\mathrm{i} = 0,\ S_\mathrm{f} = -1)\,,\\
&\gamma\,\mathrm{n} \to \Sigma^+\mathrm{K}^-, \quad (S_\mathrm{i} = 0,\ S_\mathrm{f} = -2)
\end{aligned}$$

(where the photon and leptons are assumed to have zero strangeness). These data indicate that strangeness is a symmetry in electromagnetic interactions, a conclusion reinforced by the observation of the strangeness-conserving decay

$$\Sigma^0 \to \Lambda^0\gamma \qquad (\Delta S = S_\mathrm{f} - S_\mathrm{i} = 0)\,.$$

The baryon Σ^0 has mean lifetime $\tau = 7 \times 10^{-20}\,\mathrm{s}$.

As already mentioned, the relatively long lifetimes of strange particles indicate that, except for Σ^0, they decay via weak interactions by breaking strangeness symmetry:

$$\begin{aligned}
&\Lambda^0 \to \mathrm{p}\,\pi^- && \Sigma^- \to \mathrm{n}\,\pi^-\,,\\
&\mathrm{K}^0 \to \pi^+\pi^- && \Xi^0 \to \Lambda^0\,\pi^0\,,\\
&\mathrm{K}^+ \to \mu^+\nu_\mu && \Omega^- \to \Lambda^0\,\mathrm{K}^-\,.
\end{aligned}$$

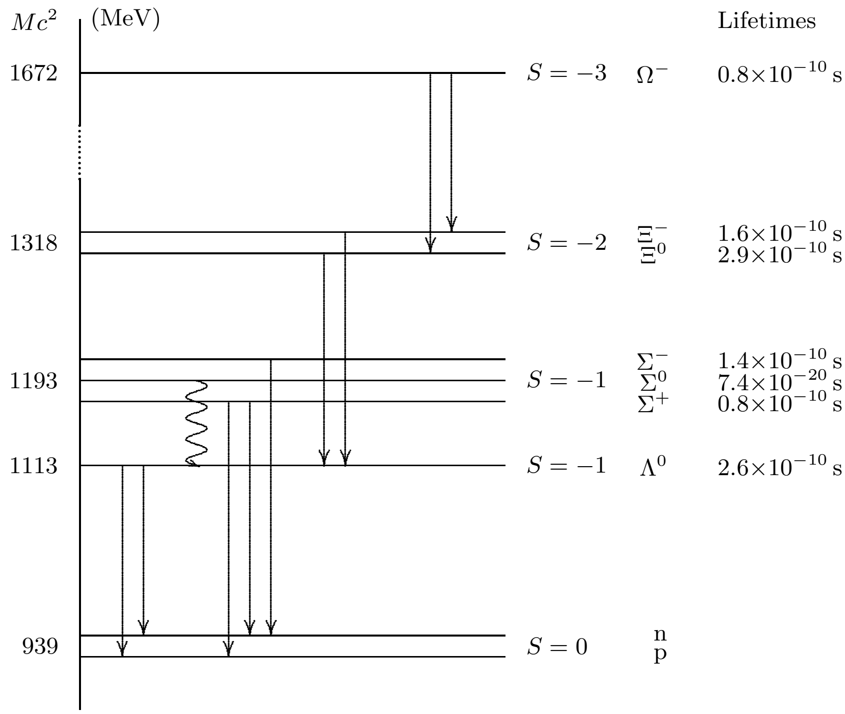

Fig. 5.3. Decay modes and lifetimes of some strange particles (a straight line represents a π meson; a wavy line, a photon)

Decays of low-lying strange particles by emission of a photon or a pion are shown in Fig. 5.3. Note that symmetry breaking obeys in all cases an extremely accurate selection rule $|\Delta S| = 1$. Transitions $|\Delta S| \geq 2$, even in a phase-space-favored decay mode such as $\Xi^- \to \mathrm{n}\,\pi^-$ for which $|\Delta S| = 2$, are either forbidden or very improbable.

To summarize, every particle is characterized by additive quantum numbers – electric charge, baryon number, lepton numbers, strangeness, as well as charm, topness (truth), bottomness (beauty) to be introduced later. The corresponding antiparticle has the same quantum numbers, but with reversed signs, and is therefore distinct from its conjugate, unless it is completely neutral. Such are the cases of the mesons π^0 (135 MeV) and η^0 (547 MeV).

5.3.2 Charge Conjugation in Field Theories

The notion of antiparticle originates from Dirac's theory of the electron. This theory predicted the existence of a particle identical to the electron except for having an electric charge with the opposite sign. This idea was substantiated by subsequent detections of the positron and other particles having the same masses and lifetimes as certain known particles, differing from them only in the signs of their respective additive quantum numbers. To relate the two types of particles it proves convenient to introduce a *unitary* operator $\mathcal{C}$ that reverses the signs of all the generalized charges of particles without affecting their spatial properties. Specifically, its action on a particle of momentum $\boldsymbol{p}$, spin s, and generalized charges, collectively represented by the symbol Q, is given by

$$\mathcal{C}\;|\boldsymbol{p}, s, Q\rangle = \xi\;|\boldsymbol{p}, s, -Q\rangle\;, \tag{5.77}$$

where ξ is a unimodular phase factor. It is not necessarily true that this operation of *charge conjugation* is identical to *field conjugation*, which replaces particles with their antiparticles. However, it turns out just to be the case for all physical particles. Therefore, $\mathcal{C}$ will be taken to represent effectively the field conjugation for all particles as well.

It follows from (77) that if $Q \neq 0$, then $[\mathcal{C}, Q] \neq 0$. In other words, a state of nonvanishing charge cannot be an eigenstate of $\mathcal{C}$. Nevertheless, the notion of charge conjugation remains useful even in these cases because of the physical consequences that follow from invariance of the system when this invariance holds. On the other hand, for a particle or system of particles completely neutral, $\mathcal{C}$ commutes with all the generalized charge generators and therefore may have common eigenstates with these operators. For such states, since $\mathcal{C}^2 = 1$, the eigenvalues of $\mathcal{C}$ are ± 1.

As the antiparticle concept is unknown in nonrelativistic quantum mechanics, the language of relativistic quantum field theories is the only one suited to its study.

Scalar Field. We may begin by defining the field conjugation operator $\mathcal{C}$ by its action on a complex scalar field

$$\mathcal{C}\,\phi(x)\,\mathcal{C}^{-1} = \xi_{\mathrm{B}}\phi^\dagger(x); \qquad \mathcal{C}^\dagger\mathcal{C} = 1, \quad |\xi_{\mathrm{B}}|^2 = 1\,. \tag{5.78}$$

We will prove that it implies (77).

It is evident that the Lagrangian for the noninteracting complex scalar field

$$\mathcal{L}_{\mathrm{B}}(x) = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \tag{5.79}$$

(where normal-ordered products are understood) is invariant to $\mathcal{C}$ because

$$\mathcal{C}\,\mathcal{L}_{\mathrm{B}}(x)\,\mathcal{C}^{-1} = |\xi_{\mathrm{B}}|^2(\partial_\mu\phi\partial^\mu\phi^\dagger - m^2\phi\phi^\dagger) = \mathcal{L}_{\mathrm{B}}(x)\,. \tag{5.80}$$

In the last step, the implicit convention of normal-ordered products has been used in permuting the field operators ϕ and $\phi^\dagger$.

The action of $\mathcal{C}$ on the Fock operators for the charged boson can be determined from (78) with ϕ replaced by its Fourier series

$$\sum_p C_{\boldsymbol{p}}[\,\mathcal{C}\,a_{\boldsymbol{p}}\,\mathcal{C}^{-1}\mathrm{e}^{-\mathrm{i}p\cdot x} + \mathcal{C}\,b^\dagger_{\boldsymbol{p}}\,\mathcal{C}^{-1}\mathrm{e}^{\mathrm{i}p\cdot x}\,] = \xi_{\mathrm{B}}\sum_p C_{\boldsymbol{p}}[\,a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x} + b_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}\,]\,.$$

Identifying the corresponding coefficients on both sides, one obtains

$$\begin{aligned}\mathcal{C}\, a_{\boldsymbol{p}}\, \mathcal{C}^{-1} &= \xi_{\mathrm{B}}\ b_{\boldsymbol{p}}\,,\\ \mathcal{C}\, b_{\boldsymbol{p}}\, \mathcal{C}^{-1} &= \xi_{\mathrm{B}}^{*} a_{\boldsymbol{p}}\,.\end{aligned} \tag{5.81}$$

For a one-particle state one gets, assuming a C-invariant vacuum, $\mathcal{C}\ |0\rangle = |0\rangle$,

$$\begin{aligned}\mathcal{C} a_{\boldsymbol{p}}^{\dagger}\ |0\rangle &= \mathcal{C}\, a_{\boldsymbol{p}}^{\dagger}\, \mathcal{C}^{-1}\mathcal{C}\ |0\rangle = \xi_{\mathrm{B}}^{*} b_{\boldsymbol{p}}^{\dagger}\ |0\rangle\ ,\\ \mathcal{C} b_{\boldsymbol{p}}^{\dagger}\ |0\rangle &= \mathcal{C}\, b_{\boldsymbol{p}}^{\dagger}\, \mathcal{C}^{-1}\mathcal{C}\ |0\rangle = \xi_{\mathrm{B}}\ a_{\boldsymbol{p}}^{\dagger}\ |0\rangle\ .\end{aligned}$$

Thus, as defined, $\mathcal{C}$ transforms a particle into its antiparticle, and vice versa, without changing their momenta.

The Noether current associated with global gauge transformations of the Lagrangian (79) is given by an expression of the form (13). The action of $\mathcal{C}$ on this current is

$$\begin{aligned}\mathcal{C}\, j_\mu(x)\, \mathcal{C}^{-1} &= \mathrm{i}\,[\phi\partial_\mu\phi^\dagger - (\partial_\mu\phi)\phi^\dagger]\\ &= -\, j_\mu(x)\,.\end{aligned} \tag{5.82}$$

The generalized charge Q defined by the space integral of $j^0(x)$ is evidently conserved. It changes sign under the C-conjugation, $\mathcal{C}\, Q\, \mathcal{C}^{-1} = -Q$. Therefore, the operation $\mathcal{C}$ defined by (78) changes the signs of electric charge, baryon number etc., as expected from the definition of charge conjugation. If the particle is completely neutral, the associated field is Hermitian, $\phi^\dagger = \phi$, and therefore $a_{\boldsymbol{p}} = b_{\boldsymbol{p}}$, and the phase becomes real, $\xi_{\mathrm{B}} = \xi_{\mathrm{B}}^{*}$. Since $\mathcal{C}^2 = 1$ and $\mathcal{C}^\dagger\mathcal{C} = 1$, it follows that $\xi_{\mathrm{B}} = \pm 1$. This *multiplicative* quantum number, when it can be defined, is called the *charge (conjugation) parity*.

Electromagnetic Field. If we assume that the transformation rule for the current density (82) is generally valid (as it proves indeed to be the case), the Maxwell field A_μ must transform according to

$$\mathcal{C}\, A_\mu(x)\, \mathcal{C}^{-1} = -A_\mu(x) \tag{5.83}$$

to generate an electromagnetic interaction invariant with respect to $\mathcal{C}$. It follows that

$$\mathcal{C}\, a(\boldsymbol{k},\lambda)\, \mathcal{C}^{-1} = -a(\boldsymbol{k},\lambda)\,. \tag{5.84}$$

Therefore, a photon state

$$\mathcal{C}\ |\boldsymbol{k},\lambda\rangle = -\ |\boldsymbol{k},\lambda\rangle \tag{5.85}$$

is also an eigenstate of $\mathcal{C}$ of eigenvalue $\xi_\gamma = -1$. The photon is *odd* under charge conjugation.

Dirac Field. Just as for a charged boson, the charge conjugate of a Dirac field must be proportional to its complex conjugate ψ^*, which suggests the following definition of the operator $\mathcal{C}$ on the Hilbert space for fermions:

$$\mathcal{C}\,\psi(x)\,\mathcal{C}^{-1} = \xi_{\mathrm{F}} B \psi^*(x), \quad |\xi_{\mathrm{F}}| = 1\,, \tag{5.86}$$

where B is a 4×4 unitary matrix on the spinor representation. Since $\overline{\psi}$ rather than ψ^* appears frequently in formulas, it is more practical to state the rule in the equivalent form

$$\mathcal{C}\,\psi(x)\,\mathcal{C}^{-1} = \xi_{\mathrm{F}} C \overline{\psi}^{\mathrm{T}}(x)\,, \quad |\xi_{\mathrm{F}}| = 1\,. \tag{5.87}$$

We have used the relation $\overline{\psi}^{\mathrm{T}} = \gamma_0^* \psi^*$ and introduced another 4×4 matrix $C = B\gamma_0^*$, which is also unitary $C^\dagger C = 1$. Note that in (87) the transposition T applies only to the spinor, not to the Fock operators. To find C it is required that Dirac's equation for ψ be covariant, or equivalently, the corresponding Lagrangian be invariant to charge conjugation. In the latter viewpoint the condition reads

$$\mathcal{C}\,\mathcal{L}_{\mathrm{F}}(x)\,\mathcal{C}^{-1} \;=\; \mathcal{L}_{\mathrm{F}}(x)\,. \tag{5.88}$$

It is then convenient to use the explicitly Hermitian version of $\mathcal{L}_{\mathrm{F}}$,

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} &\equiv \mathcal{L}_1 + \mathcal{L}_1^\dagger \\ &= \frac{1}{2}\overline{\psi}\left[\mathrm{i}\gamma^\mu \overrightarrow{\partial}_\mu - m\right]\psi + \frac{1}{2}\overline{\psi}\left[-\mathrm{i}\gamma^\mu \overleftarrow{\partial}_\mu - m\right]\psi\,. \end{aligned} \tag{5.89}$$

Noting that $\mathcal{C}\,\psi^\dagger\,\mathcal{C}^{-1} = \xi_{\mathrm{F}}^* \psi^{\mathrm{T}} \gamma_0^* C^\dagger$, one gets for $\mathcal{L}_1$,

$$\mathcal{C}\,\mathcal{L}_1\,\mathcal{C}^{-1} = \frac{1}{2}\psi^{\mathrm{T}}\gamma_0^* C^\dagger (\mathrm{i}\gamma^0\gamma^\mu\partial_\mu - \gamma_0 m) C \overline{\psi}^{\mathrm{T}}\,. \tag{5.90}$$

Since the expression on the right-hand side is a scalar, it may be equivalently replaced by its transpose in spinor space. A permutation of the anticommuting operators ψ and $\overline{\psi}$ has the effect of introducing an additional minus sign plus a c-number term given by their anticommutation rules. This c-number term drops out because $\mathcal{L}_{\mathrm{F}}$ is implicitly normal ordered, thus leaving

$$\mathcal{C}\,\mathcal{L}_1\,\mathcal{C}^{-1} = \frac{1}{2}\overline{\psi} C^{\mathrm{T}}\left(-\mathrm{i}\gamma^{\mu\mathrm{T}}\gamma^{0\mathrm{T}}\overleftarrow{\partial}_\mu + \gamma_0^{\mathrm{T}} m\right) C^* \gamma_0 \psi\,. \tag{5.91}$$

A similar calculation applies to $\mathcal{C}\,\mathcal{L}_1^\dagger\,\mathcal{C}^{-1}$. To satisfy (88) it suffices to require

$$\mathcal{C}\,\mathcal{L}_1\,\mathcal{C}^{-1} = \mathcal{L}_1^\dagger\,, \tag{5.92}$$

which implies

$$C^\dagger \gamma_\mu C = -\gamma_\mu^{\mathrm{T}} \qquad \text{(in arbitrary representation of } \gamma_\mu\text{)}\,. \tag{5.93}$$

To obtain an explicit expression for the matrix C it is useful to adopt a specific representation for the γ_μ. In the standard representation, the basic condition (93) becomes

$$C^\dagger \gamma_\mu C = -\gamma_\mu \quad (\mu = 0, 2)\,,$$
$$C^\dagger \gamma_\mu C = +\gamma_\mu \quad (\mu = 1, 3)\,,$$

which admits the solution

$$C = \lambda \gamma^2 \gamma^0\,, \quad |\lambda| = 1 \quad (\text{in standard representation of } \gamma_\mu)\,. \tag{5.94}$$

With (93) the charge conjugate of the adjoint $\overline{\psi}$ can be easily found:

$$\begin{aligned} \mathcal{C}\,\overline{\psi}\,\mathcal{C}^{-1} &= \mathcal{C}\,\psi^\dagger \gamma_0\,\mathcal{C}^{-1} = \mathcal{C}\,\psi^\dagger\,\mathcal{C}^{-1}\gamma_0 \\ &= \xi_{\mathrm{F}}^* \psi^{\mathrm{T}} \gamma_0 C^\dagger \gamma_0 = -\xi_{\mathrm{F}}^* \psi^{\mathrm{T}} \gamma_0 \gamma_0 C^\dagger \\ &= -\xi_{\mathrm{F}}^*\,\psi^{\mathrm{T}} C^\dagger\,. \end{aligned} \tag{5.95}$$

It follows that an arbitrary bilinear covariant of field operators has the transformation property

$$\mathcal{C}\,\overline{\psi}\Gamma\psi\,\mathcal{C}^{-1} = \overline{\psi}\,C\Gamma^{\mathrm{T}}C^\dagger\,\psi\,. \tag{5.96}$$

In particular,

$$\begin{aligned} C\gamma_\mu^{\mathrm{T}} C^\dagger &= -\gamma_\mu\,; \\ C\gamma_5^{\mathrm{T}} C^\dagger &= \ \gamma_5\,; \\ C(\gamma_\mu\gamma_5)^{\mathrm{T}} C^\dagger &= \ \gamma_\mu\gamma_5\,. \end{aligned}$$

Thus, the current for the Dirac particle, $j_\mu = \overline{\psi}\gamma_\mu\psi$, and the associated charge obey the expected transformation rules

$$\begin{aligned} &\mathcal{C}\,j_\mu\,\mathcal{C}^{-1} = \mathcal{C}\,\overline{\psi}\,\mathcal{C}^{-1}\gamma_\mu \mathcal{C}\,\psi\,\mathcal{C}^{-1} = -j_\mu, \\ &\mathcal{C}\,Q\,\mathcal{C}^{-1} = -Q\,. \end{aligned} \tag{5.97}$$

Next, we examine how the Fock operators and hence particle or antiparticle states transform. First, note that in the standard representation of the γ_μ and with the choice $\lambda = \mathrm{i}$, the spinors u and v, explicitly given in (3.45) and (3.46), are related through the C-matrix by

$$\begin{aligned} &C\bar{u}^{\mathrm{T}}(\boldsymbol{p}, s) = v(\boldsymbol{p}, s)\,, \\ &C\bar{v}^{\mathrm{T}}(\boldsymbol{p}, s) = u(\boldsymbol{p}, s)\,, \qquad\qquad C = \mathrm{i}\gamma^2\gamma^0\,. \end{aligned} \tag{5.98}$$

The two sides of (87) then become in terms of the Fock operators

$$\mathcal{C}\,\psi\,\mathcal{C}^{-1} = \sum_{\boldsymbol{p},s} C_{\boldsymbol{p}} \left[\,\mathcal{C}\,b(\boldsymbol{p}, s)\,\mathcal{C}^{-1} u(\boldsymbol{p}, s)\mathrm{e}^{-\mathrm{i}p\cdot x} \,+\, \mathcal{C}\,d^\dagger(\boldsymbol{p}, s)\,\mathcal{C}^{-1} v(\boldsymbol{p}, s)\mathrm{e}^{\mathrm{i}p\cdot x}\right],$$

$$\xi_{\mathrm{F}} C\overline{\psi}^{\mathrm{T}} = \xi_{\mathrm{F}} \sum_{\boldsymbol{p},s} C_{\boldsymbol{p}} \left[\,b^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s)\,\mathrm{e}^{\mathrm{i}p\cdot x} \,+\, d(\boldsymbol{p}, s) u(\boldsymbol{p}, s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\right]\,.$$

The transformation rules for the Fock operators immediately follow:

$$\begin{aligned} \mathcal{C}\, b(\boldsymbol{p}, s)\, \mathcal{C}^{-1} &= \xi_{\mathrm{F}}\, d(\boldsymbol{p}, s)\,, \\ \mathcal{C}\, d(\boldsymbol{p}, s)\, \mathcal{C}^{-1} &= \xi_{\mathrm{F}}^{*}\, b(\boldsymbol{p}, s)\,. \end{aligned} \tag{5.99}$$

They confirm a known result: field conjugation (87) converts a particle state $b^{\dagger}(\boldsymbol{p}, s)\ |0\rangle$ into the corresponding antiparticle state $\xi_{\mathrm{F}}^{*} d^{\dagger}(\boldsymbol{p}, s)\ |0\rangle$ without changing its spin or momentum. All charges however change signs according to (97). This result is illustrated in Fig. 5.4.

Finally, let us recall that a fermion and its conjugate partner have opposite parities, opposite chiralities, but equal helicities (see Problem 5.9).

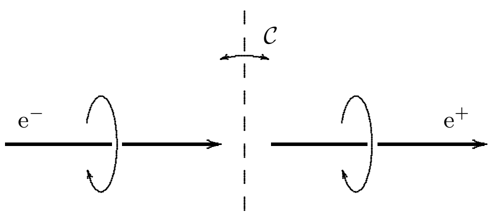

Fig. 5.4. $\mathcal{C}$ flips the sign of the charge of a particle without changing its spin or momentum

5.3.3 Interactions

For completely neutral particles, one may define a multiplicative quantum number associated with charge conjugation symmetry called the *charge (conjugation) parity*, ξ, which takes values $+1$ or -1. Examples where this concept applies are the photon, the mesons π^0 and η^0, and the self-conjugate pairs, such as e^-e^+, $p\,\bar{p}$, and $\pi^-\pi^+$.

Assuming that the **electromagnetic interaction** is C-invariant, the photon is odd ($\xi_\gamma = -1$) and an n-photon system has charge parity $(-)^n$: states of an even number of photons are even and states of an odd number of photons are odd with respect to charge conjugation, regardless of their spatial configurations. *Furry's theorem* immediately follows from this result. It says that the matrix element of an operator invariant to charge conjugation vanishes if the number of external photons is odd and if there are no other particles. This is because if there are n_{i} and n_{f} photons in the initial and final states of some given process, the total number of external photons for the process is $N = n_{\mathrm{i}} + n_{\mathrm{f}}$. Since the interaction operator is assumed to be even, conservation of charge parity requires $(-)^{n_{\mathrm{i}}} = (-)^{n_{\mathrm{f}}}$, which implies N an even integer.

From the observed two-gamma decay modes of π^0 and η^0

$$\begin{aligned} &\pi^0 \to 2\gamma \qquad \text{(branching ratio: 98.8\%)}\,, \\ &\eta^0 \to 2\gamma \qquad \text{(branching ratio: 38.8\%)}\,, \end{aligned}$$

one infers their charge parities, $\xi_{\pi^0} = 1$ and $\xi_{\eta^0} = 1$. On the other hand, the very small rates of their three-photon decay modes confirm that $\mathcal{C}$ is indeed a symmetry for electromagnetic interactions:

$$\begin{aligned} &\pi^0 \to 3\gamma \qquad (3 \times 10^{-8}) \,, \\ &\eta^0 \to 3\gamma \qquad (5 \times 10^{-4}) \,. \end{aligned}$$

The charge parity ξ also serves to identify states of particle–antiparticle systems. Consider first a state of *scalar–antiscalar* particles of relative orbital angular momentum ℓ:

$$|\phi_\ell(\mathrm{s\bar{s}})\rangle = \int \mathrm{d}^3p \, F_\ell(\boldsymbol{p}) a^\dagger_{\boldsymbol{p}} b^\dagger_{-\boldsymbol{p}} \; |0\rangle \tag{5.100}$$

($a^\dagger$, $b^\dagger$ create a boson and a conjugate antiboson, respectively). Application of $\mathcal{C}$ on both sides leads to

$$\mathcal{C} \; |\phi_\ell(\mathrm{s\bar{s}})\rangle = \int \mathrm{d}^3p \, F_\ell(\boldsymbol{p}) b^\dagger_{\boldsymbol{p}} a^\dagger_{-\boldsymbol{p}} \; |0\rangle \; . \tag{5.101}$$

After permuting the positions of the two operators (without introducing any additional signs) and changing the sign of $\boldsymbol{p}$ in the integral, one gets

$$\mathcal{C} \; |\phi_\ell(\mathrm{s\bar{s}})\rangle = \int \mathrm{d}^3p \, F_\ell(-\boldsymbol{p}) a^\dagger_{\boldsymbol{p}} b^\dagger_{-\boldsymbol{p}} \; |0\rangle \; . \tag{5.102}$$

$F_\ell(-\boldsymbol{p})$ is an eigenstate of angular momentum ℓ, and so $F_\ell(-\boldsymbol{p}) = (-)^\ell F_\ell(\boldsymbol{p})$ from (4). It follows that

$$\mathcal{C} \; |\phi_\ell(\mathrm{s\bar{s}})\rangle = (-)^\ell \; |\phi_\ell(\mathrm{s\bar{s}})\rangle \,, \tag{5.103}$$

and the charge parity of the system is $\xi(\mathrm{s\bar{s}}) = (-)^\ell$. Thus for example, the $\pi^+\pi^-$ system is even or odd with respect to $\mathcal{C}$ according to the parity of its orbital angular momentum. If in addition the boson is self-conjugate, i.e. identical to its antiparticle, ℓ may only be an even integer by Bose statistics, and $\xi(\mathrm{s\bar{s}}) = +1$ always. This is the case of a $\pi^0\pi^0$ pair, for example.

Let us consider now a pair of *fermion–antifermion* of individual spins $1/2$. Assuming the situation to be nonrelativistic, one may neglect the effects of spin-orbit coupling and of virtual fields. A state of relative orbital angular momentum ℓ and total spin $S = 0$ or 1 can then be described by

$$\left|\psi_{\ell,S}(\mathrm{f\bar{f}})\right\rangle = \int \mathrm{d}^3p \sum_{s,s'} F^{ss'}_{\ell S}(\boldsymbol{p}) b^\dagger(\boldsymbol{p}, s) d^\dagger(-\boldsymbol{p}, s') \; |0\rangle \tag{5.104}$$

($b^\dagger$ creates a fermion, $d^\dagger$ an antifermion), and its charge conjugate is

$$\mathcal{C} \; \left|\psi_{\ell,S}(\mathrm{f\bar{f}})\right\rangle = \int \mathrm{d}^3p \sum_{s,s'} F^{ss'}_{\ell S}(\boldsymbol{p}) d^\dagger(\boldsymbol{p}, s) b^\dagger(-\boldsymbol{p}, s') \; |0\rangle \; . \tag{5.105}$$

After permuting the two Fock operators, which introduces an additional minus sign to account for their anticommutation, changing the sign of momentum integration variable, and exchanging the two spin variables, one gets

$$\begin{aligned}\mathcal{C}\,\left|\psi_{\ell,S}(\mathrm{f\bar{f}})\right\rangle &= -\int \mathrm{d}^3p \sum_{s,s'} F^{s's}_{\ell S}(-\boldsymbol{p}) b^\dagger(\boldsymbol{p},s) d^\dagger(-\boldsymbol{p},s')\;|0\rangle \\ &= -(-)^{\ell+S+1}\int \mathrm{d}^3p \sum_{s,s'} F^{ss'}_{\ell S}(\boldsymbol{p}) b^\dagger(\boldsymbol{p},s) d^\dagger(-\boldsymbol{p},s')\;|0\rangle\,, \end{aligned}\tag{5.106}$$

where on the last line $F^{s's}_{\ell S}(-\boldsymbol{p}) = (-)^{\ell+S+1}F^{ss'}_{\ell S}(\boldsymbol{p})$, whose sign arises from exchanging the particle space and spin variables. Therefore, the charge parity of a self-conjugate fermion–antifermion pair is given by

$$\mathcal{C}\,\left|\psi_{\ell,S}(\mathrm{f\bar{f}})\right\rangle = (-)^{\ell+S}\,\left|\psi_{\ell,S}(\mathrm{f\bar{f}})\right\rangle\,. \tag{5.107}$$

In Table 5.2 the charge parity is given together with the ordinary parity (defined by space inversion) for an arbitrary state $n\,^{2S+1}\ell_J$ with principal quantum number n.

Table 5.2. Parities of pion–pion and fermion–antifermion systems

States	P	C	CP
$\pi^+\pi^-(\ell)$	$(-)^\ell$	$(-)^\ell$	1
$\pi^0\pi^0(\ell)$	1	1	1
$\mathrm{f\bar{f}}(n\,^{2S+1}\ell_J)$	$(-)^{\ell+1}$	$(-)^{\ell+S}$	$(-)^{S+1}$

As a first example, take the *positronium* – a bound positron–electron system – which has an energy spectrum similar to that of the hydrogen atom but with energy spacings approximately halved. The ground state is a 1^1S_0 ($\ell = 0\,,\ S = 0$) level, separated from the first excited level 1^3S_1 ($\ell = 0\,,\ S = 1$) by $8.4 \times 10^{-4}\,$eV. The charge parities are, from Table 5.2, $\xi(^1S_0) = +1$ and $\xi(^3S_1) = -1$. Since ξ is conserved in electromagnetic interaction, the singlet state $^1S_0$ decays into an even number of photons, while the triplet state $^3S_1$ decays into 3, 5, … photons (the one-real-photon mode being forbidden by energy-momentum conservation). Experiments show that the ground state 1^1S_0 decays indeed into two real photons and the excited state 1^3S_1 into three photons (at a rate reduced by a factor $\alpha = 1/137$ and, additionally, by a smaller phase space volume in the final state).

As another example, take the *mesons* treated not as elementary entities, but as bound quark–antiquark states. Neglecting the presence of gluons, we can apply the above results directly to this case. The quark and antiquark spins being $1/2$, the total spin of the quark–antiquark pair is either $S = 0$

or $S = 1$. Assuming that the lowest energy state has the most symmetric spatial configuration ($\ell = 0$), one can expect the existence of *pseudoscalar* mesons ${}^1S_0$ of parity $P = -1$ and charge parity $C = +1$, and of *vector* mesons ${}^3S_1$ with $P = -1$ and $C = -1$. We have already noted above the electromagnetic decay modes of neutral pseudoscalar mesons π^0 and η^0:

$$\text{quark–antiquark } {}^1S_0 \ (C = 1) \quad \rightarrow \quad \gamma\gamma \ (C = 1) \, .$$

But their three-photon modes are strongly suppressed:

$$\text{quark–antiquark } {}^1S_0 \ (C = 1) \quad \rightarrow \quad \gamma\gamma\gamma \ (C = -1) \, ,$$

which confirms the charge conjugation symmetry in electromagnetic interactions at this level as well. Decays of neutral vector mesons ρ (770 MeV), ω (782 MeV), or ϕ (1020 MeV) into photons alone have not been observed. But with such high energies available, it is unlikely that photons produced by the decays could exist for long without being converted into pairs of leptons or hadrons. For instance, the e^+e^- pair production observed in decays of any of these three mesons can be viewed as resulting from the quark–antiquark pair annihilation into one *virtual* photon which in turn creates a pair of relativistic positron–electron:

$$\text{quark–antiquark } {}^3S_1 \ (C = -1) \quad \rightarrow \quad \text{virtual photon} \quad \rightarrow \quad e^+e^- \, .$$

Hadronic interactions are invariant to charge conjugation, as can be verified by comparing the energy and momentum distributions of the charge conjugate reactions like

$$\pi^+ p \rightarrow \pi^+ p \quad \text{and} \quad \pi^- \bar{p} \rightarrow \pi^- \bar{p} \, ,$$

or

$$p\bar{p} \rightarrow \pi^+ + a + b \quad \text{and} \quad p\bar{p} \rightarrow \pi^- + \bar{a} + \bar{b} \, .$$

However, **weak interactions** break charge conjugation symmetry, as it was already clear from the first experiments performed in 1957 to detect parity violations. It was then established in particular that the helicities of electrons and positrons emitted in weak interactions have opposite signs. It turned out that negative and positive muons produced in weak processes also have opposite helicities. Now, the helicity operator, $\boldsymbol{\Sigma}{\cdot}\boldsymbol{p}$, is invariant to charge conjugation. Therefore, if charge conjugation is a conserving transformation, one should have

$$\begin{aligned} \left\langle e^- \,|\, \boldsymbol{\Sigma} \cdot \boldsymbol{p} \,|\, e^- \right\rangle &= \left\langle e^- \,\middle|\, \mathcal{C}^{-1}\mathcal{C} \, \boldsymbol{\Sigma}{\cdot}\boldsymbol{p} \, \mathcal{C}^{-1}\mathcal{C} \,\middle|\, e^- \right\rangle \\ &= \left\langle e^+ \,|\, \boldsymbol{\Sigma}{\cdot}\boldsymbol{p} \,|\, e^+ \right\rangle \, , \end{aligned}$$

which is contrary to observations, cf. (34).

The breakdown of charge conjugation symmetry is implicit in the empirical Hamiltonian for β-decay (37). It has already been experimentally established that the four coupling parameters contained in this Hamiltonian,

$C_{\rm V}$, $C_{\rm A}$, $C'_{\rm V} = \alpha_{\rm V} C_{\rm V}$, and $C'_{\rm A} = \alpha_{\rm A} C_{\rm A}$ are all relatively real. Now charge conjugation invariance of this Hamiltonian would imply $C_{\rm V}$ and $C_{\rm A}$ are relatively real, as are $C'_{\rm V}$ and $C'_{\rm A}$, but that $C_{\rm V}$ and $C'_{\rm V}$ differ by a relative phase of $90°$, as do $C_{\rm A}$ and $C'_{\rm A}$. This contradicts observations.

Let us examine in more detail the decay Hamiltonian $\mathcal{H}_\beta$ given in (32), keeping only the V and A terms. To simplify notations, we denote $\psi_{\rm p}$ by p and so on, and write the β-decay Hamiltonian as

$$\begin{aligned}\mathcal{H}_\beta = & C_{\rm V}(\bar{p}\gamma_\mu n)(\bar{e}\gamma^\mu \nu) + C'_{\rm V}(\bar{p}\gamma_\mu n)(\bar{e}\gamma_5\gamma^\mu \nu) \\ & + C_{\rm A}(\bar{p}\gamma_\mu\gamma_5 n)(\bar{e}\gamma^\mu\gamma_5 \nu) - C'_{\rm A}(\bar{p}\gamma_\mu\gamma_5 n)(\bar{e}\gamma^\mu \nu)\,, \end{aligned} \tag{5.108}$$

and obtain its Hermitian conjugate

$$\begin{aligned}\mathcal{H}^\dagger_\beta = & C^*_{\rm V}(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma^\mu e) + C'^*_{\rm V}(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma_5\gamma^\mu e) \\ & + C^*_{\rm A}(\bar{n}\gamma_\mu\gamma_5 p)(\bar{\nu}\gamma^\mu\gamma_5 e) - C'^*_{\rm A}(\bar{n}\gamma_\mu\gamma_5 p)(\bar{\nu}\gamma^\mu e)\,, \end{aligned} \tag{5.109}$$

and its charge conjugate

$$\begin{aligned}\mathcal{C}\,\mathcal{H}_\beta\,\mathcal{C}^{-1} = & C_{\rm V}(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma^\mu e) - C'_{\rm V}(\bar{n}\gamma_\mu p)(\bar{\nu}\gamma_5\gamma^\mu e) \\ & + C_{\rm A}(\bar{n}\gamma_\mu\gamma_5 p)(\bar{\nu}\gamma^\mu\gamma_5 e) + C'_{\rm A}(\bar{n}\gamma_\mu\gamma_5 p)(\bar{\nu}\gamma^\mu e)\,. \end{aligned} \tag{5.110}$$

Invariance of the complete Hamiltonian $\mathcal{H}_\beta + \mathcal{H}^\dagger_\beta$ implies $\mathcal{C}\,\mathcal{H}_\beta\,\mathcal{C}^{-1} = \mathcal{H}^\dagger_\beta$, or the condition that $C_{\rm V}$ and $C_{\rm A}$ be both real, and $C'_{\rm V}$ and $C'_{\rm A}$ be both imaginary. Therefore, charge conjugation symmetry is violated. It is interesting to note however that if one now applies the parity operation $\mathcal{P}$ on both sides of (110), the only effect on the right-hand side is to flip the signs of the $C'_{\rm V}$ and $C'_{\rm A}$ terms. Therefore, comparison with $\mathcal{H}^\dagger_\beta$ tells us invariance of the full Hamiltonian under combined $\mathcal{P}$ and $\mathcal{C}$ requires all C_i, C'_i for $i = $ V, A to be relatively real, in agreement with observations.

To summarize the results of this and previous sections, the weak processes described by the Hamiltonian $\mathcal{H}_\beta + \mathcal{H}^\dagger_\beta$ break parity and charge conjugation symmetries separately, but are invariant to $\mathcal{T}$, to the combined operation $\mathcal{CP}$, and evidently also to $\mathcal{CPT}$. This example illustrates the general situation to be treated in the following section.

5.4 The CPT Theorem

This theorem, due to Lüders, Pauli, and Schwinger, states that the product of the transformations $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ applied in any order is always a symmetry of a quantum theory if the Lagrangian that defines it is Hermitian, invariant under proper Lorentz transformations, is built up from normal-ordered products of fields, and if the fields are quantized in accord with the usual spin–statistics connection.

The operation $\Theta \equiv \mathcal{CPT}$ is defined as the product of the transformations $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ performed in any order. From the definitions of $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$, the action of Θ on various variables can be found. Namely:

– All complex numbers are replaced by their complex conjugates;
– Space-time coordinates x^μ are replaced by $x^{\mu\prime} = -x^\mu$;
– Scalar fields are transformed as

$$\Theta\,\phi(x)\,\Theta^{-1} = \omega_{\mathrm{B}}\phi^\dagger(-x)\,,$$

where the phase should be *fixed* at $\omega_{\mathrm{B}} = +1$ in order to realize invariance of interactions involving scalar fields;
– The electromagnetic field transforms as

$$\Theta\,A_\mu(x)\,\Theta^{-1} = -A_\mu(-x)\,;$$

– Fermion fields transform as

$$\Theta\,\psi(x)\,\Theta^{-1} = \omega_{\mathrm{F}}\gamma_5\gamma_0\overline{\psi}^{\mathrm{T}}(-x), \quad |\omega_{\mathrm{F}}| = 1\,;$$

– Bilinear products of fermion fields of the form $\overline{\psi}\Gamma\psi$ may be grouped according to their transformations under Θ: $\Gamma_+ = \{1, \sigma^{\mu\nu}, \mathrm{i}\gamma_5\}$ are even (do not change signs), and $\Gamma_- = \{\gamma^\mu, \gamma^\mu\gamma_5\}$ are odd (change signs).

A Lorentz-invariant quantity is constructed for example by multiplying a Γ_- by another Γ_-, $A^\mu(x)$, or ∂/∂_μ, or by multiplying Γ_+ by a factor having the same even number of Lorentz indices, then by contracting all repeated Lorentz indices. In a Θ-transformation, individual factors involving fermion fields may change their signs or phases, but their products do not change signs or phases. For example, in the familiar β-decay,

$$\Theta\,[\mathcal{H}_\beta(x) + \mathcal{H}^\dagger_\beta(x)]\,\Theta^{-1} = \mathcal{H}_\beta(-x) + \mathcal{H}^\dagger_\beta(-x)\,.$$

We have seen that a product of quantum fields representing a physical quantity should always be normal-ordered. When it is necessary to restore the field factors to such an order, no sign change is needed when permuting Bose fields, but a minus sign is introduced for each permutation of two Fermi fields. This is why the usual connection between spins and statistics is required in the CPT theorem. When the theory contains interactions between bosons, as in $\lambda\phi^3 + \lambda^*\phi^{\dagger 3}$, or interactions between a boson field and a fermion field, as in $\mathrm{i}g\overline{\psi}\gamma_5\psi\phi$ or in $g\overline{\psi}\gamma_5\gamma^\mu\psi\partial_\mu\phi$, the transformation rules given above, with the selected phase $\omega_{\mathrm{B}} = +1$, guarantee invariance of these interactions to Θ.

In summary, the CPT theorem requires that the Lagrangian density of any physical quantum field theory transforms as

$$\Theta\,\mathcal{L}(x)\,\Theta^{-1} = \mathcal{L}(-x)^\dagger\,. \tag{5.111}$$

Since $\mathcal{L}$ is a Hermitian operator and the action function is given by the space-time integral $\int \mathrm{d}^4x\,\mathcal{L}$, the theorem guarantees invariance of the action and hence that of the theory itself. The validity of the theorem is based on the invariance to the group of continuous Lorentz transformations, the usual spin–statistics connection and the locality of the theory. It is not affected by whether $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ separately are symmetries or not.

5.4.1 Implications of CPT Invariance

Using the same method as in the preceding sections, one can show that Θ transforms a one-particle state into an antiparticle state (up to a phase factor)

$$\Theta \, |a\rangle = |\bar{a}\rangle \, e^{i\vartheta} \,. \tag{5.112}$$

Therefore, in an invariant theory,

$$\langle a \,|\, H \,|\, a\rangle = \left\langle a \,\middle|\, \Theta^{-1} H \Theta \,\middle|\, a\right\rangle = \langle \bar{a} \,|\, H \,|\, \bar{a}\rangle \,. \tag{5.113}$$

Here H is the total Hamiltonian. The result says that the energy spectra in the original and the transformed systems are identical. In particular, in the absence of interactions, $\langle a \,|\, H \,|\, a\rangle$ gives essentially the mass of the particle, and therefore CPT invariance implies the equality between the masses of the particle and the corresponding antiparticle. This result is experimentally well verified. For example, the proton and the antiproton differ in mass by

$$(m_{\mathrm{p}} - m_{\bar{\mathrm{p}}})/m_{\mathrm{p}} \approx 2 \times 10^{-11} \,.$$

The best test of CPT invariance comes from comparing the K^0 and $\bar{K}^0$ masses. We will see in Chap. 11 that the K^0–$\bar{\mathrm{K}}^0$ mass difference is related to certain CP violation parameters. The best available values for these yield

$$|(m_{\bar{\mathrm{K}}^0} \, - \, m_{\mathrm{K}^0})/m_{\mathrm{K}^0}| \leq 9 \times 10^{-19}. \tag{5.114}$$

The transition rate from state $|a\rangle$ to state $|b\rangle$ due to a weak interaction H_{F} is given by

$$w_{ba} = 2\pi |\, \langle b \,|\, H_{\mathrm{F}} \,|\, a\rangle \,|^2 \rho_b , \tag{5.115}$$

where ρ_b is the final state density. The total transition rate, obtained by summing over all possible final channels, $w_a = \sum_b w_{ba}$, is related to the total lifetime of $|a\rangle$ by $\tau_a = 1/w_a$. From invariance of H_{F} to Θ it follows that the total lifetime is also invariant:

$$\tau_a = \tau_{\bar{a}} \,. \tag{5.116}$$

A particle and its antiparticle stable to strong and electromagnetic interactions have equal lifetimes. This agrees with observations. For example,

$$(\tau_{\pi^+} - \tau_{\pi^-})/\tfrac{1}{2}\,(\tau_{\pi^+} + \tau_{\pi^-}) = (6 \pm 7) \times 10^{-4} \,,$$
$$(\tau_{\mathrm{K}^+} - \tau_{\mathrm{K}^-})/\tfrac{1}{2}\,(\tau_{\mathrm{K}^+} + \tau_{\mathrm{K}^-}) = (0.11 \pm 0.09) \times 10^{-2} \,.$$

Finally, Θ-invariance being assumed, if for some interaction one of the transformations $\mathcal{C}$, $\mathcal{P}$, or $\mathcal{T}$ is nonconserving, at least another is also nonconserving. We have already seen that in β-decay, neither $\mathcal{P}$ nor $\mathcal{C}$ is a symmetry, but $\mathcal{T}$ and $\mathcal{PC}$ are both symmetries. On the other hand, if for example neither $\mathcal{T}$ nor $\mathcal{CP}$ is a symmetry for a certain interaction, then either $\mathcal{P}$, but not $\mathcal{C}$, is conserving; or $\mathcal{C}$ is, but not $\mathcal{P}$. An interaction that breaks CP-invariance also breaks T-invariance. An example of this situation is the decay of neutral kaons to be studied in Chap. 11.

5.4.2 C, P, T, and CPT

For reference, we summarize in these paragraphs the conjugation rules for Dirac fermion fields and their bilinear products. We use the standard representation of the matrices γ_μ.
(a) Hermitian conjugation

$$\left(\overline{\psi}_1(x)\Gamma\psi_2(x)\right)^\dagger = \overline{\psi}_2(x)\gamma_0\Gamma^\dagger\gamma_0\psi_1(x)\,. \tag{5.117}$$

(b) Parity $\mathcal{P}$

$$\mathcal{P}\,\psi(x)\,\mathcal{P}^{-1} = \eta\gamma_0\psi(\tilde{x}), \quad \tilde{x} = (t, -\boldsymbol{x}), \tag{5.118}$$
$$\mathcal{P}\,\overline{\psi}(x)\,\mathcal{P}^{-1} = \eta^*\overline{\psi}(\tilde{x})\gamma_0, \tag{5.119}$$

$$\mathcal{P}\,\overline{\psi}_1(x)\Gamma\psi_2(x)\,\mathcal{P}^{-1} = \eta_1^*\eta_2\overline{\psi}_1(\tilde{x})\,\gamma_0\Gamma\gamma_0\,\psi_2(\tilde{x})\,. \tag{5.120}$$

(c) Charge conjugation $\mathcal{C}$ $\quad (C \equiv \mathrm{i}\gamma^2\gamma^0)$

$$\begin{aligned} \mathcal{C}\,\psi(x)\,\mathcal{C}^{-1} &= \xi C\overline{\psi}^{\mathrm{T}}(x), \\ \mathcal{C}\,\overline{\psi}(x)\,\mathcal{C}^{-1} &= -\xi^*\psi^{\mathrm{T}}(x)C^\dagger, \end{aligned} \tag{5.121}$$

$$\mathcal{C}\,\overline{\psi}_1(x)\Gamma\psi_2(x)\,\mathcal{C}^{-1} = \xi_1^*\xi_2\overline{\psi}_2(x)\,C\Gamma^{\mathrm{T}}C^\dagger\,\psi_1(x)\,. \tag{5.122}$$

(d) Time reversal $\mathcal{T}$ $\quad (A \equiv \gamma^1\gamma^3)$

$$\mathcal{T}\,\psi(x)\,\mathcal{T}^{-1} = \zeta A\psi(x'), \quad x' = (-t, \boldsymbol{x})\,, \tag{5.123}$$
$$\mathcal{T}\,\overline{\psi}(x)\,\mathcal{T}^{-1} = \zeta^*\overline{\psi}(x')\gamma_0A^\dagger\gamma_0\,, \tag{5.124}$$

$$\mathcal{T}\,\overline{\psi}_1(x)\Gamma\psi_2(x)\,\mathcal{T}^{-1} = \zeta_1^*\zeta_2\overline{\psi}_1(x')\,A\Gamma^*A^\dagger\,\psi_(x')\,. \tag{5.125}$$

(e) $\mathcal{CP}$ transformation

$$\mathcal{CP}\psi(x)\mathcal{P}^{-1}\mathcal{C}^{-1} = \xi\eta C\gamma_0\overline{\psi}^{\mathrm{T}}(\tilde{x})\,, \tag{5.126}$$
$$\mathcal{CP}\overline{\psi}(x)\mathcal{P}^{-1}\mathcal{C}^{-1} = -\xi^*\eta^*\psi^{\mathrm{T}}(\tilde{x})C^\dagger\gamma_0\,, \tag{5.127}$$

$$\mathcal{CP}\overline{\psi}_1(x)\Gamma\psi_2(x)\mathcal{P}^{-1}\mathcal{C}^{-1} = -\xi_1^*\xi_2\eta_1^*\eta_2\overline{\psi}_2(\tilde{x})\,\gamma^2\Gamma^{\mathrm{T}}\gamma^2\,\psi_1(\tilde{x})\,. \tag{5.128}$$

(f) $\Theta = \mathcal{CPT}$

$$\Theta\,\psi(x)\,\Theta^{-1} = \omega\gamma_5\gamma_0\overline{\psi}^{\mathrm{T}}(-x), \quad -x = (-t, -\boldsymbol{x})\,, \tag{5.129}$$
$$\Theta\,\overline{\psi}(x)\,\Theta^{-1} = \omega^*\psi^{\mathrm{T}}(-x)\gamma_5\gamma_0\,, \tag{5.130}$$

$$\Theta\,\overline{\psi}_1(x)\Gamma\psi_2(x)\,\Theta^{-1} = \omega_1^*\omega_2\overline{\psi}_2(-x)\,\gamma_5\Gamma\gamma_5\,\psi_1(-x)\,. \tag{5.131}$$

Finally, Table 5.3 gives a list of transformation properties of the basic spinor operators in the discrete symmetries discussed in this chapter.

Table 5.3. Transformations of the Γs in discrete symmetries

		$\mathcal{P}$	$\mathcal{C}$	$\mathcal{T}$	$\mathcal{CP}$	$\mathcal{CPT}$
	Γ	$\gamma_0\Gamma\gamma_0$	$C\Gamma^{\mathrm{T}}C^\dagger$	$A\Gamma^* A^\dagger$	$-\gamma^2\Gamma^{\mathrm{T}}\gamma^2$	$\gamma_5\Gamma\gamma_5$
S	1	1	1	1	1	1
P	$\mathrm{i}\gamma_5$	$-\mathrm{i}\gamma_5$	$\mathrm{i}\gamma_5$	$-\mathrm{i}\gamma_5$	$-\mathrm{i}\gamma_5$	$\mathrm{i}\gamma_5$
V	γ^μ	γ_μ	$-\gamma^\mu$	γ_μ	$-\gamma_\mu$	$-\gamma^\mu$
A	$\gamma^\mu\gamma_5$	$-\gamma_\mu\gamma_5$	$\gamma^\mu\gamma_5$	$\gamma_\mu\gamma_5$	$-\gamma_\mu\gamma_5$	$-\gamma^\mu\gamma_5$
T	$\sigma^{\mu\nu}$	$\sigma_{\mu\nu}$	$-\sigma^{\mu\nu}$	$-\sigma_{\mu\nu}$	$-\sigma_{\mu\nu}$	$\sigma^{\mu\nu}$

Remark: The position of the Lorentz index is important, e.g. $\gamma_\mu = (\gamma^0, -\gamma^i)$.

Problems

5.1 Symmetries in quantum mechanics. (a) Consider π–p scattering or more generally scattering of a spin-0 particle by a spin-½ particle. The relevant variables are $\boldsymbol{p}_{\mathrm{i}}$(relative initial momentum), $\boldsymbol{p}_{\mathrm{f}}$ (relative final momentum), $\boldsymbol{n} = \boldsymbol{p}_{\mathrm{i}} \times \boldsymbol{p}_{\mathrm{f}}/|\boldsymbol{p}_{\mathrm{i}} \times \boldsymbol{p}_{\mathrm{f}}|$ and $\boldsymbol{\sigma}$ (fermion spin). What is the general rotational invariant form of the transition amplitude? What are the restrictions when $\mathcal{P}$ or $\mathcal{T}$ invariances are imposed?
(b) Repeat the analysis for the scattering of two spin-½ particles.

5.2 Lepton decay. The coupling constant for $\mu \to \mathrm{e}\nu\bar{\nu}$ is $G_{\mathrm{F}} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$. From dimensional analysis the decay rate is proportional to $G_{\mathrm{F}}^2 m_\mu^5$. Derive its exact formula

$$\Gamma(\mu \to \mathrm{e}\nu\bar{\nu}) = \frac{G_{\mathrm{F}}^2 m_\mu^5}{192\pi^3}.$$

Give an estimate of the muon mean lifetime. In 1975 the τ lepton of mass 1.78 GeV was detected. Give estimates of the decay rates for $\tau \to \mathrm{e}\nu\bar{\nu}$ and $\tau \to \mu\nu\bar{\nu}$ and the corresponding branching ratios.

5.3 $\pi^\pm$ decays. The weak decays $\pi \to \mathrm{e}\nu$ and $\pi \to \mu\nu$ may be described by the V–A interactions

$$\mathcal{H}_{\mathrm{VA}}(x) = \frac{G_{\mathrm{F}}}{\sqrt{2}} J^\alpha_{\mathrm{hadr}}(x)\,[\bar{\ell}\gamma_\alpha(1-\gamma_5)\nu] + \mathrm{h.c.}\,, \quad \ell = \mathrm{e}, \mu\,.$$

The hadronic matrix $\langle 0\,|\,J^\alpha_{\mathrm{hadr}}\,|\,\pi\rangle$ can depend only on the four-vector $p_\pi = p_\ell + p_\nu$, and by virtue of the (Dirac) equations of motion for the leptons, the decay amplitude is proportional to the charged lepton mass. Show that

$$\frac{\Gamma(\pi \to \mu\nu)}{\Gamma(\pi \to \mathrm{e}\nu)} = P\,\frac{m_\mu^2}{m_{\mathrm{e}}^2}, \quad \text{where } P = (m_\pi^2 - m_{\mathrm{e}}^2)^2/(m_\pi^2 - m_\mu^2)^2\,.$$

5.4 The $\tau-\theta$ puzzle. (a) Consider the decay mode $\theta \to \pi^+\pi^0$. Assuming parity invariance and 0 for the spin of θ, find the parity of θ.
(b) Now consider the decay process $\tau \to \pi^+\pi^+\pi^-$. (This τ is an old symbol for the K meson.) Let ℓ be the orbital angular momentum of $\pi^+\pi^+$, and ℓ' the orbital angular momentum of π^- relative to the center-of-mass of $\pi^+\pi^+$. Assuming parity invariance and the spin of τ equal to 0, find its parity.

5.5 Λ^0 decay. The weak decay $\Lambda^0 \to \mathrm{p} + \pi^-$ can be described by the Hamiltonian

$$H_{\mathrm{int}} = \int_0^t \mathrm{d}t' \int \mathrm{d}^3x\, \phi^\dagger \overline{\psi}_{\mathrm{p}}(g + g'\gamma_5)\psi_\Lambda\ + \mathrm{h.c.}\,,$$

where ψ_Λ destroys a Λ and creates a $\bar{\Lambda}$, $\overline{\psi}_p$ creates a proton and destroys an antiproton, and $\phi^\dagger$ creates a π^- and destroys a π^+ . The h.c. term makes $\mathcal{H}_{\mathrm{int}}$ Hermitian. The initial and final states in the process $\Lambda^0 \to \mathrm{p} + \pi^-$ are given by $|\mathrm{i}\rangle = b^\dagger_\Lambda(\boldsymbol{p}, s)\ |0\rangle$, and $|\mathrm{f}\rangle = a^\dagger(\boldsymbol{k}) b^\dagger_{\mathrm{p}}(\boldsymbol{p}', s')\ |0\rangle$.
(a) Calculate the transition amplitude $\langle \mathrm{f}\,|\, H_{\mathrm{int}}\,|\,\mathrm{i}\rangle$ for Λ at rest.
(b) Suppose that Λ is polarized with spin oriented in the positive z direction, show that the relative probabilities for observing protons produced with polarizations $\pm 1/2$ are

$$\begin{aligned} |a_{\mathrm{s}} + a_{\mathrm{p}} \cos\theta|^2 &\qquad \text{for spin } + 1/2\,, \\ |a_{\mathrm{p}} \sin\theta|^2 &\qquad \text{for spin } - 1/2\,, \end{aligned}$$

where $\cos\theta = p'_z/|\boldsymbol{p}'|$. Calculate a_{s}, a_{p} .

5.6 Lepton number. Discuss how to set up an experiment to decide whether the lepton number is additive or multiplicative.

5.7 Weyl representation. Find the time inversion matrix A and the charge conjugation matrix C in the Weyl representation of the γ_μ matrices.

5.8 Quantum numbers for an antifermion. Let ψ be a Dirac wave function. Its parity η, helicity h, and chirality λ (in the ultra-relativistic limit) are defined respectively by $\mathcal{P}\psi = \eta\gamma_0\psi$, $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}\,\psi = h\psi$, and $\gamma_5\psi = \lambda\psi$. Show that the corresponding quantum numbers for the charge conjugate field $\psi^{\mathrm{c}} = \xi C\overline{\psi}^{\mathrm{T}}$ are $\eta^{\mathrm{c}} = -\eta$, $h^{\mathrm{c}} = h$, $\lambda^{\mathrm{c}} = -\lambda$.

5.9 Invariance of the electromagnetic interaction. Let A_μ and $F_{\mu\nu}$ be the electromagnetic field and field tensor. Study the transformation properties of the following interaction terms under the operations $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{PCT}$: (a) $e\overline{\psi}\gamma_\mu\psi\, A^\mu$; (b) $1/2\mu_{\mathrm{m}} F_{\mu\nu}\overline{\psi}\,\sigma^{\mu\nu}\psi$; (c) $1/2\mu_{\mathrm{e}} F_{\mu\nu}\overline{\psi}\,\gamma_5\sigma^{\mu\nu}\psi$.

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