3 Fermion Fields

In the previous chapter, we have seen that the nonrelativistic equation of motion of a free particle can be generalized in a natural way to the relativistic regime by making homogeneous its dependence on space and time. There are two possibilities. The first, involving second-order derivatives and called the Klein–Gordon equation, governs the evolution of fields of integral spins which are associated with operators that obey commutation relations. The second, containing only first-order derivatives and discovered by P. A. M. Dirac in 1928 in his search for a relativistic formalism admitting a non-negative probability density, describes the dynamics of fields having spins $1/2$. These fields must then represent particles, such as the electron, the proton, or the quarks, that constitute the bulk of visible matter in the universe. They are the subject of the present chapter.

3.1 The Dirac Equation

The equation in question is of the form

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0\,, \tag{3.1}$$

where the parameter m, of the dimension of mass, can be chosen to be real (by redefining if necessary the phase of the complex function ψ), but the four quantities $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ remain in general complex and behave, by assumption, as the components of some Lorentz vector. In particular, $\gamma_\mu = g_{\mu\nu}\gamma^\nu$. Application of the operator $(\mathrm{i}\gamma^\mu\partial_\mu + m)$ on this equation gives

$$-(\gamma^\mu\gamma^\nu\,\partial_\mu\partial_\nu + m^2)\psi \;=\; -\left[\tfrac{1}{2}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\,\partial_\mu\partial_\nu + m^2\right]\psi \;=\; 0\,. \tag{3.2}$$

Comparing the operator on the left-hand side of this equation with the energy-momentum relation for a free particle of mass m,

$$p^\mu p_\mu - m^2 = 0\,,$$

considered as an operator by the correspondence $p^\mu \to \mathrm{i}\partial^\mu$, identifies m in (1) as the particle mass and leads to the condition

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} \;=\; 2g_{\mu\nu}\,1\,. \tag{3.3}$$

The notation $\{a, b\} = ab + ba$ stands for an *anticommutator*, and 1 for a unit matrix of a certain order since γ^μ are not necessarily simple numbers. From (3) we see that, first $\gamma_0^2 = 1$ and $\gamma_i^2 = -1$ and hence the eigenvalues of γ_0 are ± 1 and those of γ_i are $\pm\sqrt{-1}$, and second, $\gamma_0 = \gamma_i \gamma_0 \gamma_i$ and $\gamma_i = -\gamma_0 \gamma_i \gamma_0$ for $i = 1, 2, 3$. Taking the traces of the last two identities, one gets

$$\begin{aligned} \mathrm{Tr}\gamma_0 &= \ \mathrm{Tr}\,(\gamma_i \gamma_0 \gamma_i) = -\mathrm{Tr}\gamma_0\,, \\ \mathrm{Tr}\gamma_i &= -\mathrm{Tr}\,(\gamma_0 \gamma_i \gamma_0) = -\mathrm{Tr}\gamma_i\,, \end{aligned}$$

where the cyclic property of trace has been used: $\mathrm{Tr}(abc) = \mathrm{Tr}(cab)$. So $\mathrm{Tr}\gamma_\mu = 0$. Since the trace of a matrix is equal to the sum of its eigenvalues, γ_0 must have as many eigenvalues equal to $+1$ as those equal to -1, and similarly for γ_i. It follows that the order of the matrix γ_μ must be an even number. The smallest possible order, $N = 2$, is not admissible because it has just enough room for the three Pauli matrices and the unit matrix. The next smallest order for which the γ_μ matrices can be realized according to (3) is $N = 4$, which can accommodate 16 independent matrices, and it is this case that interests us. In this representation (called the *spinor representation*), ψ is a column vector with four components (called the *Dirac spinor*) and the γ_μ are 4×4 complex matrices. The mass m will be assumed to be nonvanishing for now. The special case $m = 0$, for which the representation $N = 2$ is perfectly appropriate, will be reconsidered at the end of this chapter. Let us note in passing that the equality of the dimensions of the spinor representation and of space-time is a pure coincidence that occurs only in four-dimensional space-time.

Let us rewrite (1) in the form

$$\mathrm{i}\gamma^0 \frac{\partial}{\partial t}\psi \;=\; (-\mathrm{i}\boldsymbol{\gamma}\cdot\nabla + m)\psi\,.$$

Multiplying both sides by γ^0 yields a relativistic version of the Schrödinger equation for the system

$$\mathrm{i}\frac{\partial}{\partial t}\psi \;=\; (-\mathrm{i}\gamma^0\boldsymbol{\gamma}\cdot\nabla + m\gamma^0)\psi\,. \tag{3.4}$$

The operator on the right-hand side may be identified with the Hamiltonian of the Dirac particle

$$\hat{H} \;=\; (-\mathrm{i}\gamma^0\boldsymbol{\gamma}\cdot\nabla + m\gamma^0)\,, \tag{3.5}$$

the Hermitian conjugate of which is

$$\hat{H}^\dagger \;=\; (-\mathrm{i}\nabla\cdot\boldsymbol{\gamma}^\dagger\gamma^{0\dagger} + m\gamma^{0\dagger})\,. \tag{3.6}$$

Being an observable, $\hat{H}$ must be Hermitian, $\hat{H} = \hat{H}^\dagger$, which implies

$$\gamma_0^\dagger = \gamma_0\,, \qquad\quad \gamma_i^\dagger = \gamma_0\gamma_i\gamma_0 = -\gamma_i\,,$$

or more concisely

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \,. \tag{3.7}$$

The basic properties of the γ_μ given in (3) and (7) should suffice to define γ_μ. Nevertheless, it is sometimes useful to have an explicit matrix representation. In the most popular representation (which therefore is called the *standard representation* or the *Dirac–Pauli representation*) γ^0 is diagonal:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{3.8}$$

Here 1 is the 2×2 unit matrix and σ^i the usual 2×2 Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.9}$$

Products of the form $\gamma^\mu A_\mu$, which occur frequently in calculations involving the Dirac particles, deserve a distinctive symbol: $\not{A} \equiv \gamma^\mu A_\mu$; for example,

$$\not{\partial} = \gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{\partial t} + \boldsymbol{\gamma} \cdot \nabla \,.$$

With this notation the Dirac equation assumes the form

$$(\mathrm{i}\,\not{\partial} - m)\psi(x) \; = 0 \,. \tag{3.10}$$

The Hermitian conjugation of this equation followed by an application of rule (7) yields

$$-(\mathrm{i}\,\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^0 + m\psi^\dagger) = 0 \,,$$

which reduces after multiplication from the right by γ^0 to a simpler form

$$\overline{\psi}(\mathrm{i}\,\overleftarrow{\not{\partial}} + m) = 0 \,. \tag{3.11}$$

Here $\overline{\psi}\,\overleftarrow{\not{\partial}} \equiv \partial_\mu \overline{\psi} \gamma^\mu$, and $\overline{\psi} \equiv \psi^\dagger \gamma_0$ is a row vector with four components (called the *Hermitian adjoint spinor*). If $\psi_1, \ldots, \psi_4$ are the components of the column vector ψ, the adjoint spinor $\overline{\psi}$ is given in the representation (8) of the γ_μ by the row vector $(\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$, with ψ_a^* standing for complex conjugates of ψ_a.

Let us finally note that in the Dirac equation, just as in the Schrödinger equation, the time evolution is determined by a first-order time derivative. In both cases, if the wave function is known at time $t = 0$, it is also known at any later time $t > 0$. In contrast, to solve an equation of second-order time derivative, such as the Klein–Gordon equation, one needs to know both the wave function and its time derivative at the initial point.

3.2 Lorentz Symmetry

In this section, three important results will be derived from the Lorentz symmetry of the theory:

- the covariance of the Dirac equation, without which the theory would not be viable;
- the spin of the Dirac field;
- the bilinear covariants in ψ, a result essential to model building.

3.2.1 Covariance of the Dirac Equation

To say that the Dirac equation is covariant means that first, if an observer $\mathcal{O}$ provided with the coordinates x describes a field by $\psi(x)$ as a solution of (10), then another Lorentz observer $\mathcal{O}'$ describes the same physical field by $\psi'(x')$ that satisfies an equation of the same form written in coordinates x' of $\mathcal{O}'$, and second, that there is a well-defined relation between $\psi(x)$ and $\psi'(x')$.

A Lorentz transformation is defined by the real parameters $a^{\mu}{}_{\nu}$:

$$\begin{aligned} &x^{\mu} \to x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \,, \\ &a^{\mu}{}_{\rho}\, g_{\mu\nu}\, a^{\nu}{}_{\sigma} = g_{\rho\sigma} \,. \end{aligned} \tag{3.12}$$

Since the Dirac equation and the Lorentz transformation are both linear relations, $\psi(x)$ and $\psi'(x')$ must also be connected by a linear relation, that is, each component $\psi'_a(x')$ $(a = 1, 2, 3, 4)$ can be written as a linear combination of the components $\psi_b(x)$:

$$\psi'_a(x') \;=\; S_{ab}(a)\, \psi_b(x) \,, \tag{3.13}$$

[cf. the simpler transformation law for vector fields: $A'^{\mu}(x') = a^{\mu}{}_{\nu}\, A^{\nu}(x)$]. The 4×4 matrix S, which depends on the parameters $a^{\mu}{}_{\nu}$, is determined by requiring $\psi'(x')$ to be a solution to

$$(\mathrm{i}\gamma^{\mu}\partial'_{\mu} - m)\psi'(x') = 0 \,. \tag{3.14}$$

Multiplying (10) from the left by S, considered as a linear operator, and inserting $S^{-1}S = 1$, one obtains

$$(\mathrm{i}S\gamma^{\mu}S^{-1}\partial_{\mu} - m)\psi'(x') = 0 \,, \tag{3.15}$$

which exactly coincides with (14) provided that $S\gamma^{\mu}S^{-1}\partial_{\mu} \;=\; \gamma^{\mu}\partial'_{\mu}$. Since $\partial_{\mu} = a^{\nu}{}_{\mu}\partial'_{\nu}$, this condition means

$$S^{-1}(a)\gamma^{\mu}S(a) \;=\; a^{\mu}{}_{\nu}\gamma^{\nu} \,. \tag{3.16}$$

This is precisely what is meant when we say that γ^{μ} behaves as a Lorentz vector.

The relation (16) holds for any Lorentz transformation parameterized by $a^\mu{}_\nu$. Now we use it to determine $S(a)$ for a proper Lorentz transformation. In this case it suffices to consider an infinitesimal deviation from the identity

$$a^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu\,, \quad \text{where} \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}\,. \tag{3.17}$$

To first order in $\epsilon_{\mu\nu}$, $S(a)$ must have the general form

$$S(a) \approx 1 - \frac{\mathrm{i}}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}\,, \tag{3.18}$$

where $\sigma_{\mu\nu}$ are 4×4 matrices antisymmetric in their Lorentz indices, $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$, and the numerical factor $-\mathrm{i}/4$ has been introduced by convention. The condition (16) then becomes to first order in $\epsilon_{\mu\nu}$,

$$\epsilon^\nu{}_\mu\gamma^\mu = -\frac{\mathrm{i}}{4}\epsilon^{\kappa\lambda}(\gamma^\nu\sigma_{\kappa\lambda} - \sigma_{\kappa\lambda}\gamma^\nu)\,,$$

which reduces to

$$2\mathrm{i}(\delta^\nu{}_\kappa\gamma_\lambda - \delta^\nu{}_\lambda\gamma_\kappa) = [\gamma^\nu\,,\,\sigma_{\kappa\lambda}]\,.$$

As solution to this equation, one finds with the help of (3)

$$\sigma_{\mu\nu} \;=\; \frac{\mathrm{i}}{2}\,[\gamma_\mu, \gamma_\nu]\,. \tag{3.19}$$

Its Hermitian conjugate is

$$\sigma^\dagger_{\mu\nu} = \gamma_0\sigma_{\mu\nu}\gamma_0\,, \tag{3.20}$$

or explicitly,

$$\sigma^\dagger_{ij} = \sigma_{ij}\,, \qquad \sigma^\dagger_{0i} = -\sigma_{0i}\,, \tag{3.21}$$

where the following identities have been used: $[\gamma_0, \sigma_{ij}] = 0\,;\{\gamma_0, \sigma_{0i}\} = 0$. The six independent matrices $\sigma^{\mu\nu}$ are $\sigma^{0j} = \mathrm{i}\gamma^0\gamma^j$ and $\sigma^{ij} = \mathrm{i}\gamma^i\gamma^j$ for $i \neq j$. Accordingly, we introduce the useful notations α^i and Σ^i, which are given in the standard representation by

$$\sigma^{0j} \equiv \mathrm{i}\alpha^j \;=\; \mathrm{i}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}\,,$$

$$\sigma^{ij} \equiv \epsilon^{ijk}\,\Sigma^k \;=\; \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\,. \tag{3.22}$$

An important property of S follows from (20), namely,

$$S^\dagger = \gamma_0\,S^{-1}\,\gamma_0\,, \tag{3.23}$$

which also shows that in general S is not unitary. The Lorentz transforms of the Hermitian conjugate spinor and of the adjoint spinor are

$$\begin{aligned}
\psi'(x')^\dagger &= \psi^\dagger(x) S^\dagger\,, \\
\overline{\psi}'(x')^\dagger &= \psi^\dagger(x) S^\dagger \gamma^0 \;=\; \overline{\psi}(x) S^{-1}\,.
\end{aligned} \tag{3.24}$$

They show that bilinear products of the form $\overline{\psi}\Gamma\psi$ in general transform more simply than $\psi^\dagger\Gamma\psi$. As $S(a)$ will play a central role in what immediately follows, let us consider a few examples.

Example 3.1 Rotation ($\hat{z}, \delta\theta$)
An infinitesimal rotation about the z axis through an angle $\delta\theta$ is defined by the parameter $\epsilon_{12} = -\delta\theta$, and the corresponding rotation matrix for the spinor is

$$S_{\mathrm{R}}(\hat{z}, \delta\theta) \approx 1 - \frac{\mathrm{i}}{4}\epsilon_{ij}\,\sigma^{ij} = 1 + \frac{\mathrm{i}}{2}\delta\theta\,\sigma^{12}\,. \tag{3.25}$$

More generally, for the finite rotation through an angle θ about an arbitrary axis $\hat{\boldsymbol{n}}$, the rotation matrix for the spinor is obtained by replicating (25):

$$\begin{aligned}
S_{\mathrm{R}}(\hat{\boldsymbol{n}}, \theta) &= \exp\left(\frac{\mathrm{i}}{2}\,\theta\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}\right) \\
&= \cos\frac{\theta}{2} + \mathrm{i}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}\,\sin\frac{\theta}{2}\,.
\end{aligned} \tag{3.26}$$

To obtain the second line, the exponential has first been expanded in a series in powers of $\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}$, then even and odd powers have been summed up separately using $(\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma})^2 = (\hat{\boldsymbol{n}})^2 = 1$. Hermiticity of σ^{ij} (or Σ^i) then implies the unitarity of S in this case: $S_{\mathrm{R}}^\dagger = S_{\mathrm{R}}^{-1}$.

Example 3.2 Lorentz Boost
An infinitesimal Lorentz boost in the x direction is defined by the parameter $\epsilon_{01} = -\delta\omega$, and the corresponding transformation matrix for the spinor is

$$S_{\mathrm{L}}(\delta\omega) = 1 - \frac{1}{2}\,\delta\omega\,\alpha^1\,.$$

For an arbitrary finite boost $\omega\hat{\boldsymbol{n}}$ the matrix reads

$$\begin{aligned}
S_{\mathrm{L}}(\omega\hat{\boldsymbol{n}}) &= \exp\left(-\tfrac{1}{2}\,\omega\hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha}\right) \\
&= \cosh\frac{\omega}{2} - \hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha}\,\sinh\frac{\omega}{2}\,.
\end{aligned}$$

To obtain the second line, it is useful to note $(\hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha})^2 = (\hat{\boldsymbol{n}})^2 = 1$. As the matrices $\alpha^j = -\mathrm{i}\sigma^{0j}$ are Hermitian, S_{L} is also Hermitian, $S_{\mathrm{L}}^\dagger = S_{\mathrm{L}}$, rather than unitary. ■

3.2.2 Spin of the Dirac Field

Just as for a vector field (see Chap. 2), we proceed first by determining the total angular momentum of the spinor field through the examination of its property (13),

$$\psi'(x') = S(a)\psi(x)$$

or

$$\psi'(x) = S(a)\psi(a^{-1}x)\,. \tag{3.27}$$

To the first order of an infinitesimal transformation, $a^\mu{}_\nu \approx \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$, and the transformed field is

$$\begin{aligned}\psi'(x) &= \left(1 - \tfrac{\mathrm{i}}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}\right)\,\psi(x^\rho - \epsilon^\rho{}_\sigma x^\sigma)\\ &= \left(1 - \tfrac{\mathrm{i}}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}\right)\,\left[\psi(x) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu}L_{\mu\nu}\,\psi(x)\right]\\ &= \psi(x) - \tfrac{\mathrm{i}}{2}\epsilon^{\mu\nu}\,\left(L_{\mu\nu} + \tfrac{1}{2}\,\sigma_{\mu\nu}\right)\,\psi(x)\,,\end{aligned}$$

which implies the field variation

$$\delta_0\psi(x) = -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu}J_{\mu\nu}\,\psi(x)\,. \tag{3.28}$$

Let us write for once this relation with spinor labels explicitly shown:

$$\delta_0\psi_a(x) = -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu}\left(J_{\mu\nu}\right)_{ab}\,\psi_b(x)\,. \tag{3.29}$$

Here $J_{\mu\nu}$ are the generators for infinitesimal Lorentz transformations,

$$\left(J_{\mu\nu}\right)_{ab} = L_{\mu\nu}\delta_{ab} + \tfrac{1}{2}\,\left(\sigma_{\mu\nu}\right)_{ab}\,, \tag{3.30}$$

where $\sigma_{\mu\nu}$, defined by (19), corresponds to the intrinsic part of the transformation, and $L_{\mu\nu}$, defined in the previous chapter, to the orbital contribution,

$$L_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)\,.$$

When the transformation being applied is a pure rotation, the corresponding generators are of course simply the angular momentum components

$$J^k = \tfrac{1}{2}\,\epsilon^{ijk}\left(L_{ij} + \tfrac{1}{2}\,\sigma_{ij}\right) \equiv L^k + \tfrac{1}{2}\,\Sigma^k. \tag{3.31}$$

The square of the spin $\frac{1}{2}\boldsymbol{\Sigma}$ is

$$\tfrac{1}{2}\,\boldsymbol{\Sigma}\cdot\tfrac{1}{2}\,\boldsymbol{\Sigma} = (\tfrac{1}{2}\,\boldsymbol{\sigma})\cdot(\tfrac{1}{2}\,\boldsymbol{\sigma}) = \tfrac{3}{4} = \tfrac{1}{2}\,(1+\tfrac{1}{2}\,)\,,$$

which shows that the field described by the Dirac equation has spin $s = 1/2$.

3.2.3 Bilinear Covariants

The presence of half-angles in Lorentz rotations, such as in (26), implies that a rotation through an angle of 4π or a multiple of 4π is needed to bring a spinor $\psi(x)$ back to its initial value. Therefore, physical observables in the Dirac theory must involve combinations of even powers of $\psi(x)$, the simplest being of the second power.

A simple example is provided by the current density. It can be derived by multiplying (10) from the left by $\overline{\psi}$ and (11) from the right by ψ, and by summing up the resulting expressions:

$$\overline{\psi}(\mathrm{i}\gamma\cdot\partial+\mathrm{i}\gamma\cdot\overleftarrow{\partial})\psi=\mathrm{i}\partial_\mu(\overline{\psi}\gamma^\mu\psi)\;=\;0\,.$$

The result has the form of a conservation law

$$\partial_\mu j^\mu(x)=0\,, \tag{3.32}$$

and immediately suggests the definition of a current for the free Dirac field

$$j^\mu(x)=\overline{\psi}(x)\gamma^\mu\psi(x)\,. \tag{3.33}$$

The zero-component, $j^0=\overline{\psi}(x)\gamma^0\psi(x)=\psi^\dagger\psi=|\psi_1|^2+|\psi_2|^2+|\psi_3|^2+|\psi_4|^2$, can be interpreted as a probability density; it is real and positive, exactly the property required. The current density j^μ behaves as a Lorentz vector, because using (24),

$$\begin{aligned}j'^\mu(x')&=\overline{\psi}'(x')\gamma^\mu\psi'(x')\\&=\overline{\psi}(x)S^{-1}\gamma^\mu S\psi(x)\\&=a^\mu{}_\nu\overline{\psi}(x)\gamma^\nu\psi(x)\,.\end{aligned}$$

Then, since j^μ is a vector, the divergence $\partial_\mu j^\mu$ is a Lorentz-invariant and the continuity equation (32) itself is invariant.

In view of applications to come, it is useful to determine now all the basic bilinear covariants of the form $\overline{\psi}\Gamma\psi$, of which $j^\mu=\overline{\psi}\gamma^\mu\psi$ is but an example. Since the dimension of the spinor representation is 4, there must exist 16 linearly independent 4×4 matrices, which can be constructed from products of 0, 1, 2, 3, and 4 γ-matrices. They are

$$\begin{aligned}&\Gamma_{\mathrm{S}}=\;1\quad, &&\Gamma^\mu_{\mathrm{V}}=\gamma^\mu\,, &&\Gamma^{\mu\nu}_{\mathrm{T}}=\sigma^{\mu\nu}\,,\\&\Gamma^\mu_{\mathrm{A}}=\gamma_5\gamma^\mu\,, &&\Gamma_{\mathrm{P}}=\mathrm{i}\gamma_5.\end{aligned} \tag{3.34}$$

Here we have introduced a new symbol

$$\begin{aligned}\gamma_5=\gamma^5\;&=\;\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\\&=\frac{\mathrm{i}}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,.\end{aligned} \tag{3.35}$$

In a general Lorentz transformation γ_5 obeys the relation

$$S^{-1}\gamma_5 S \;=\; S^{-1}(\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3)S \;=\; (\det\, a)\,\gamma_5\,, \tag{3.36}$$

and so is invariant to a proper transformation ($\det\, a = +1$) but changes sign in an inversion or a reflection ($\det\, a = -1$). Such transformation properties are characteristic of *pseudoscalar* quantities. For similar reasons, $\gamma_5\gamma^\mu$ transforms as an *axial vector*. In the standard representation of the γ-matrices, γ_5 is given by

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.37}$$

The bilinear covariants associated with the Γs represent various couplings the Dirac fields may have with themselves or with other fields. With the Γs so chosen, they are real (Problem 3.3), independent from one another, and have the characteristic Lorentz transformation properties shown in Table 3.1.

Table 3.1. Bilinear covariants

Representations	Γ	Lorentz transformations
Scalar	1	$\bar\psi'(x')\psi'(x') = \bar\psi(x)\psi(x)$
Pseudoscalar	$\mathrm{i}\gamma_5$	$\bar\psi'(x')\mathrm{i}\gamma_5\psi'(x') = \det[a]\,\bar\psi(x)\mathrm{i}\gamma_5\psi(x)$
Vector	γ^μ	$\bar\psi'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\,\bar\psi(x)\gamma^\nu\psi(x)$
Axial vector	$\gamma_5\gamma^\mu$	$\bar\psi'(x')\gamma_5\gamma^\mu\psi'(x') = \det[a]\,a^\mu{}_\nu\,\bar\psi(x)\gamma_5\gamma^\nu\psi(x)$
Tensor	$\sigma^{\mu\nu}$	$\bar\psi'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\alpha\,a^\nu{}_\beta\,\bar\psi(x)\sigma^{\alpha\beta}\psi(x)$

3.3 Free-Particle Solutions

A plane-wave solution to the Dirac equation (10) is

$$\psi(x) = w(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,, \tag{3.38}$$

where the coefficient $w(p)$ is an x-independent spinor the four components of which satisfy a system of homogeneous equations

$$(\not{p} - m)_{ab}\,w_b(p) = 0\,. \tag{3.39}$$

For a nontrivial solution to the latter equations to exist, it is necessary that $\det(\not{p} - m) = 0$, which together with (8) gives $m^2 + \boldsymbol{p}^2 - p_0^2 = 0$. This characteristic equation yields two possible eigenvalues for the energy, $p^0 = \pm E$, with $E = \sqrt{\boldsymbol{p}^2 + m^2}$, to which correspond the wave functions

$$\psi_\pm(x) = \begin{cases} u(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}p\cdot x}\,, \\ v(\boldsymbol{p})\mathrm{e}^{+\mathrm{i}p\cdot x}\,, \end{cases} \tag{3.40}$$

where now $p^\mu = (E, \boldsymbol{p})$, with $p^0 = E > 0$ in both cases. The stationary spinors satisfy the equations

$$(\not{p} - m)\, u(\boldsymbol{p}) = 0\,, \tag{3.41}$$

$$(\not{p} + m)\, v(\boldsymbol{p}) = 0\,. \tag{3.42}$$

The spinor $u(\boldsymbol{p})$ will be referred to as the *positive-energy* solution, and $v(\boldsymbol{p})$ as the *negative-energy* solution. The corresponding adjoint spinors, $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$, obey the equations

$$\bar{u}(\boldsymbol{p})(\not{p} - m) = 0\,, \qquad \bar{v}(\boldsymbol{p})(\not{p} + m) = 0\,. \tag{3.43}$$

As spinors play an essential role in the study of the Dirac particles, it is important to examine in detail their properties. To facilitate the arguments, it will be useful to adopt the standard representation of the γ-matrices. Therefore, some results, such as the explicit form of the spinors, depend on the specific representation chosen, but the final expressions for the observables should be independent of the representation.

3.3.1 Normalized Spinors

In the rest frame of the particle ($\boldsymbol{p} = 0$) equation (41) reads

$$m(\gamma^0 - 1)u(\mathbf{0}) = -2m \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u(\mathbf{0}) = 0\,.$$

Here the matrix elements 0 and 1 are themselves 2×2 matrices. Writing the Dirac four-component spinor u in terms of two two-component Pauli spinors ξ and η,

$$u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}\,,$$

the above equation yields $\eta = 0$ and two degenerate solutions for ξ.

For $\boldsymbol{p} \neq 0$ (41) may be cast in the form

$$(E - m)\xi - \boldsymbol{\sigma}\cdot\boldsymbol{p}\,\eta = 0\,,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\xi - (E + m)\eta = 0\,.$$

The solution obtained for the Pauli spinor,

$$\eta = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\,\xi\,, \tag{3.44}$$

then leads to two independent solutions

$$u(\boldsymbol{p},s) = N\begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\,\chi_s \end{pmatrix}, \qquad (s=1,2), \tag{3.45}$$

where N is a normalization factor. The Pauli spinors χ_s, for $s=1,2$, are linearly independent and may be normalized according to $\chi_s^\dagger\chi_{s'} = \delta_{ss'}$.

The solutions to (42) can be similarly found:

$$v(\boldsymbol{p},s) = N'\begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\,\eta_s \\ \eta_s \end{pmatrix}, \qquad (s=1,2), \tag{3.46}$$

where η_s are two normalized Pauli spinors, $\eta_s^\dagger\eta_{s'} = \delta_{ss'}$.

The spinors χ_s and η_s may be chosen for example as the eigenvectors of the spin operator $^1\!/_2\,\sigma^3$,

$$\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_2 = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \eta_1 = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \eta_2 = \begin{pmatrix}-1\\0\end{pmatrix}. \tag{3.47}$$

They are related by

$$\eta_s = -\mathrm{i}\sigma^2\,\chi_s = (-)^{(1-2s)/2}\chi_{-s} \qquad \text{for } s = \pm\tfrac{1}{2}\,. \tag{3.48}$$

In this choice the subscripts s correspond to the eigenvalues of $^1\!/_2\,\sigma^3$. The phases of η_s are fixed so that the corresponding charge conjugate spinors (to be introduced in Chap. 5) can be more simply defined. It is sometimes useful to make the spin eigenvalues explicit in the labels according to the conversion rules

$$\chi_1 = \chi_{1/2}\,, \quad \chi_2 = \chi_{-1/2}\,, \qquad \eta_1 = \eta_{1/2}\,, \quad \eta_2 = \eta_{-1/2}\,. \tag{3.49}$$

They correspond up to a normalization constant to the Dirac spinors

$$u(\mathbf{0},1) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \; u(\mathbf{0},2) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \; v(\mathbf{0},1) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \; v(\mathbf{0},2) = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}. \tag{3.50}$$

In conclusion, the Dirac equation has four independent solutions $u(\boldsymbol{p},s)$ and $v(\boldsymbol{p},s)$, with $s=1$, 2 representing positive-energy and negative-energy polarization states of a spin-$^1\!/_2$ field.

3.3.2 Completeness Relations

To fix the normalization constant of the spinors, let us first calculate the scalar product

$$
\begin{aligned}
u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p},s') &= N^2 \begin{pmatrix} 1 & \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}/(E+m) \end{pmatrix} \chi_s^\dagger \chi_{s'} \\
&= N^2 \left(1 + \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{(E+m)^2} \right) \delta_{ss'} \; = N^2 \frac{2E}{E+m}\, \delta_{ss'}\,.
\end{aligned}
$$

Since $u^\dagger u = \bar{u}\gamma^0 u$ behaves as the time component of a Lorentz vector, the normalization factor N must be chosen so that the right-hand side of the last equation has the same property,

$$
u^\dagger(\boldsymbol{p},s)\, u(\boldsymbol{p},s') \;=\; 2E\, \delta_{ss'}\,, \tag{3.51}
$$

with a numerical factor fixed for convenience. This implies $N = \sqrt{E+m}\,$. By the same token, with the normalization $N' = \sqrt{E+m}$ the noncovariant norm of v is given by

$$
v^\dagger(\boldsymbol{p},s)\, v(\boldsymbol{p},s') \;=\; 2E\, \delta_{ss'}\,. \tag{3.52}
$$

Note however that $v^\dagger(\boldsymbol{p},-s)u(\boldsymbol{p},s) \neq 0$ but in contrast $v^\dagger(-\boldsymbol{p},s')\,u(\boldsymbol{p},s) = 0$ and $u^\dagger(\boldsymbol{p},s)v(-\boldsymbol{p},s') = 0$. The reason is that $u(\boldsymbol{p},s)$ and $v(-\boldsymbol{p},s)$, rather than $v(\boldsymbol{p},s)$, are the eigenspinors of the Dirac Hamiltonian

$$
H_{\boldsymbol{p}} = \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)\,, \tag{3.53}
$$

with respective eigenvalues E and $-E$:

$$
\begin{aligned}
H_{\boldsymbol{p}} u(\boldsymbol{p}) &= \;\; E\, u(\boldsymbol{p})\,, \\
H_{\boldsymbol{p}} v(-\boldsymbol{p}) &= -E\, v(-\boldsymbol{p})\,,
\end{aligned} \tag{3.54}
$$

and are therefore mutually orthogonal. The four spinors $u(\boldsymbol{p},s)$ and $v(-\boldsymbol{p},s)$ for $s = 1,2$ form a complete set in the spinor representation for any given $\boldsymbol{p}$, which implies the closure relation

$$
\sum_{s=1}^{2} \Big[u_a(\boldsymbol{p},s)u_b^\dagger(\boldsymbol{p},s) + v_a(-\boldsymbol{p},s)v_b^\dagger(-\boldsymbol{p},s) \Big] = 2E\, \delta_{ab}\,, \tag{3.55}
$$

where $a,\, b = 1,\ldots,4$ label the spinor components.

The above results, written in terms of the Hermitian conjugates of spinors, would be better re-expressed in terms of the spinor adjoint conjugates because the latter are just those involved in bilinear covariants. In general, in order to make explicit calculations of bilinear covariants, one has to use

Dirac's equations (41)–(42). Let us for example evaluate the scalar product $\bar{u}(\boldsymbol{p},s)\,u(\boldsymbol{p},s')$. Multiplying (41) from the left by $u^\dagger(\boldsymbol{p},s)$ gives

$$u^\dagger(\boldsymbol{p},s)(\gamma\cdot p-m)\,u(\boldsymbol{p},s')=u^\dagger(\boldsymbol{p},s)(\gamma^0E-\boldsymbol{\gamma}\cdot\boldsymbol{p}-m)\,u(\boldsymbol{p},s')=0. \quad (3.56)$$

Its Hermitian conjugate, with the spin indices s, s' interchanged, reads

$$u^\dagger(\boldsymbol{p},s)(\gamma^\dagger\cdot p-m)\,u(\boldsymbol{p},s')=u^\dagger(\boldsymbol{p},s)(\gamma^0E+\boldsymbol{\gamma}\cdot\boldsymbol{p}-m)\,u(\boldsymbol{p},s')=0. \quad (3.57)$$

Summing the last two equations gives

$$Eu^\dagger(\boldsymbol{p},s)\gamma^0u(\boldsymbol{p},s')=mu^\dagger(\boldsymbol{p},s)\,u(\boldsymbol{p},s')=2Em\,\delta_{ss'}\,,$$

by making use of (51). The other three invariant products are calculated in the same way, with the help of (41)–(42), (51)–(52), and $\gamma_0^\dagger=\gamma_0$, $\gamma_i^\dagger=-\gamma_i$, leading to the results

$$\begin{aligned}\bar{u}(\boldsymbol{p},s)\,u(\boldsymbol{p},s')&=\ \ 2m\,\delta_{ss'}\,,\\ \bar{v}(\boldsymbol{p},s)\,v(\boldsymbol{p},s')&=-2m\,\delta_{ss'}\,,\\ \bar{u}(\boldsymbol{p},s)\,v(\boldsymbol{p},s')&=\ \bar{v}(\boldsymbol{p},s)\,u(\boldsymbol{p},s')\ =\ 0\,.\end{aligned} \quad (3.58)$$

The norms defined in this way are covariant. Note also that the norm of the negative-energy spinor v is negative; the sign difference with the corresponding noncovariant norm can be traced to the minus sign in the γ^0 matrix that comes with $\bar{v}=v^\dagger\gamma^0$.

To rewrite the completeness relation (55) in terms of $\bar{u}$ and $\bar{v}$, first note that, with (54),

$$\begin{aligned}(H_{\boldsymbol{p}}+E)\,u(\boldsymbol{p},s)&=\ \ 2E\,u(\boldsymbol{p},s)\,,\\ (H_{\boldsymbol{p}}-E)\,v(-\boldsymbol{p},s)&=-2E\,v(-\boldsymbol{p},s)\,;\end{aligned} \quad (3.59)$$

that is, $H_{\boldsymbol{p}}\pm E$ act as projection operators: $H_{\boldsymbol{p}}-E$ cancels $u(\boldsymbol{p})$ but leaves $v(-\boldsymbol{p})$ essentially unchanged, whereas $H_{\boldsymbol{p}}+E$ cancels $v(-\boldsymbol{p})$ but leaves $u(\boldsymbol{p})$ unchanged. However, these operators are not in a covariant form, an inconvenience in an invariant theory, and so should be replaced by equivalent operators which are. For this purpose, it suffices to note that

$$\begin{aligned}(\not{p}-m)\,u&=0\,,\\ (\not{p}+m)\,u&=2m\,u\,,\end{aligned}$$

and also

$$\begin{aligned}(\not{p}+m)\,v&=0\,,\\ (\not{p}-m)\,v&=-2m\,v\,.\end{aligned}$$

Then it is clear that $\not{p}\pm m$ are precisely the operators we are seeking. It is also useful to have them re-expressed in terms of spinors, which requires somewhat more work. When $H_{\boldsymbol{p}} + E$ is applied on both sides of (55), one gets $\sum uu^\dagger$ on the left-hand side and $H_{\boldsymbol{p}} + E$ on the right-hand side:

$$\sum_s u(\boldsymbol{p}, s)u^\dagger(\boldsymbol{p}, s) = H_{\boldsymbol{p}} + E = \gamma^0(\boldsymbol{\gamma} \cdot \boldsymbol{p} + m) + E\,. \tag{3.60}$$

The desired result is obtained by multiplying both sides from the right by γ^0:

$$\sum_s u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s) = \gamma^0 E - \boldsymbol{\gamma} \cdot \boldsymbol{p} + m = \gamma_\mu p^\mu + m\,. \tag{3.61}$$

Proceeding in the same manner with $H_{\boldsymbol{p}} - E$, one gets

$$\sum_s v(-\boldsymbol{p}, s)\bar{v}(-\boldsymbol{p}, s) = \gamma^0 E + \boldsymbol{\gamma} \cdot \boldsymbol{p} - m\,, \tag{3.62}$$

which, after reversing the sign of $\boldsymbol{p}$ on both sides, leads to the result

$$\sum_s v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s) = \gamma^0 E - \boldsymbol{\gamma} \cdot \boldsymbol{p} - m = \gamma_\mu p^\mu - m\,. \tag{3.63}$$

It is useful to introduce next the operators

$$\begin{aligned}
\Lambda_+(p) &\equiv \frac{\not{p} + m}{2m} = \frac{1}{2m}\sum_s u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s)\,, \\
\Lambda_-(p) &\equiv \frac{-\not{p} + m}{2m} = -\frac{1}{2m}\sum_s v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s)\,.
\end{aligned} \tag{3.64}$$

Applied on an arbitrary spinor the operator $\Lambda_+(p)$ gives the positive-energy components, while $\Lambda_-(p)$ projects out the negative-energy components. They are therefore the projection operators for the positive-energy solutions and the negative-energy solutions, respectively. Their basic properties are summarized in the following relations, valid for any given momentum,

$$\begin{aligned}
\Lambda_\pm^2 &= \Lambda_\pm\,, \\
\Lambda_+\Lambda_- &= \Lambda_-\Lambda_+ = 0\,, \\
\Lambda_+ + \Lambda_- &= 1\,.
\end{aligned} \tag{3.65}$$

The last of these relations is just the covariant form of the closure relation exactly equivalent to (55):

$$\sum_s [\,u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s) \;-\; v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s)\,] \;=\; 2m\,. \tag{3.66}$$

3.3.3 Helicities

In this subsection we examine the exact physical significance of the two degrees of freedom s for each of the positive-energy or negative-energy solutions of a free particle. For a particle at rest, $\boldsymbol{p} = 0$, it is clear that the spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ can be constructed as eigenspinors of the spin operator

$$\tfrac{1}{2}\,\Sigma^3 = \tfrac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix},$$

with $s = 1, 2$ associated with the eigenvalues $\pm 1/2$. It suffices then to choose the spinors as in (50). However, in general, for any nonvanishing momentum vector $\boldsymbol{p}$ not lying in the z direction, the free-particle solutions given in (45) and (46) are not eigenvectors of Σ^3. In other words, let χ_λ be the two-component spinors that are eigenfunctions of the Hermitian matrix $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$,

$$\tfrac{1}{2}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi_\lambda = \lambda\,\chi_\lambda\,. \tag{3.67}$$

Then the solutions to the Dirac equation for a free particle in (45) and (46), but with the Pauli spinors χ_s and η_s both replaced by χ_λ, are the eigenspinors of $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$ for $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$. This follows from the simple fact that $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$ commutes with itself.

In two-component spinor space, it is even possible to diagonalize $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ for an *arbitrary* unit vector $\hat{\boldsymbol{n}}$. However, as this operator does not generally commute with $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$ [which appears in the Dirac spinors (45) and (46)], it is not possible to construct the solutions to the Dirac equation for a free particle as four-component eigenspinors of $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}$ for an arbitrary $\hat{\boldsymbol{n}}$, unless $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{p}}$ or $\hat{\boldsymbol{n}} = 0$. Indeed, with the Hamiltonian defined in (53) and $\boldsymbol{\Sigma}$ considered as a Heisenberg operator, the Heisenberg equation for $\boldsymbol{\Sigma}$ reads

$$\frac{\mathrm{d}\boldsymbol{\Sigma}}{\mathrm{d}t} = \mathrm{i}\,[H_{\boldsymbol{p}}, \boldsymbol{\Sigma}] = -2(\boldsymbol{\alpha}\times\boldsymbol{p})\,. \tag{3.68}$$

As in general $\boldsymbol{\alpha}\times\boldsymbol{p} \neq 0$, it follows that $\mathrm{d}\boldsymbol{\Sigma}/\mathrm{d}t \neq 0$ and the spin $\boldsymbol{\Sigma}$ is not a constant vector (although the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\,\boldsymbol{\Sigma}$ of course is). Now forming the dot product of both sides of the above equation with some constant vector $\hat{\boldsymbol{n}}$,

$$\frac{\mathrm{d}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}}{\mathrm{d}t} = -2(\boldsymbol{\alpha}\times\boldsymbol{p})\cdot\hat{\boldsymbol{n}}\,, \tag{3.69}$$

it is seen that $\mathrm{d}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}/\mathrm{d}t \neq 0$, unless $\hat{\boldsymbol{n}} = 0$ or $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{p}}$.

As $\boldsymbol{L}\cdot\hat{\boldsymbol{p}} = 0$, one has $\boldsymbol{J}\cdot\hat{\boldsymbol{p}} = \frac{1}{2}\,\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$. The operator $\boldsymbol{J}\cdot\hat{\boldsymbol{p}}$ or $\frac{1}{2}\,\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$ is called the *helicity operator* for a spin-$1/2$ particle. One refers to the eigenstates of helicity $h = +1/2$ as the *right-handed states* (with spin oriented in the

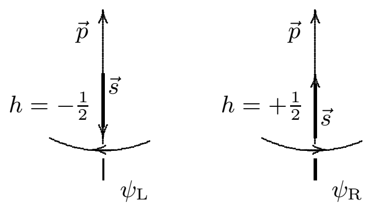

Fig. 3.1. Relative orientation of spin and momentum for left-handed and right-handed particles

direction of motion), and to those of helicity $h = -1/2$ as the *left-handed states* (with spin opposite to the direction of motion). See Fig. 3.1.

Given an arbitrary spinor, how can we extract its component having a specified (circular) polarization? We expect that the operators that perform this task are some covariant generalizations of an operator found in nonrelativistic quantum mechanics,

$$P(\hat{\boldsymbol{n}}) = \tfrac{1}{2}\,(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\,,$$

which projects out of a given Pauli spinor the component polarized in the $\hat{\boldsymbol{n}}$ direction. The operators we are seeking will separate states $s = 1$ and $s = 2$, just as $\Lambda_\pm$ separate the spinors u and v. They should be orthonormalized, should have the correct nonrelativistic limit, and finally, should commute with both $\Lambda_\pm$. The operators that satisfy these conditions are

$$P(\pm n) = \tfrac{1}{2}\,(1 \pm \gamma_5\, \not{n})\,, \tag{3.70}$$

where n^μ is a normalized spacelike vector, $n_\mu n^\mu = -1$ (to satisfy the first two conditions), and is orthogonal to the particle momentum, $n_\mu p^\mu = 0$ (to satisfy the last condition).

For a system at rest, $\boldsymbol{p} = 0$, the condition $n_\mu p^\mu = 0$ implies $n^0 = 0$, and so to have $n \cdot n = -1$, it suffices to orient n^μ in the z direction, so that $n^\mu = (0, 0, 0, 1)$. Then, in the standard representation of the γ-matrices,

$$P(\pm n) = \tfrac{1}{2}\,(1 \mp \gamma_5\gamma^3) = \tfrac{1}{2}\begin{pmatrix} 1 \pm \sigma^3 & 0 \\ 0 & 1 \mp \sigma^3 \end{pmatrix}. \tag{3.71}$$

With the spinors χ_s and η_s in (45) and (46) chosen as eigenspinors of σ^3, the operators $P(\pm n)$ perform the required tasks:

$$\begin{aligned} P(+n)\, u(\mathbf{0}, 1) &= u(\mathbf{0}, 1)\,, \\ P(-n)\, u(\mathbf{0}, 2) &= u(\mathbf{0}, 2)\,, \\ P(+n)\, v(\mathbf{0}, 1) &= v(\mathbf{0}, 1)\,, \\ P(-n)\, v(\mathbf{0}, 2) &= v(\mathbf{0}, 2)\,, \end{aligned} \tag{3.72}$$

and cancel the other three spinors in each case. The operator $P(+n)$ projects out the state with the polarization $1/2$ in the rest frame of the particle for a positive-energy solution (right-handed particle), and $-1/2$ for a negative-energy solution (left-handed antiparticle). Similarly $P(-n)$, *mutatis mutandis*. Since (70) is Lorentz-invariant and gives the correct solution in a particular Lorentz frame, it is the required operator valid in every Lorentz frame.

When $\boldsymbol{p} \neq \mathbf{0}$, one may choose $n = n_p$ such that $\boldsymbol{n}_p$ is parallel to $\boldsymbol{p}$:

$$n_p = \left(\frac{|\boldsymbol{p}|}{m}, \frac{p^0}{m} \hat{\boldsymbol{p}} \right) . \tag{3.73}$$

With this choice, the polarization becomes identical to the helicity. This can be seen for example by matrix multiplication, using the standard representation of the γ, as follows:

$$\begin{aligned}(1 + \gamma_5 \not{n}_p)\,(\pm \not{p} + m) &= (1 \pm \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}})\,(\pm \not{p} + m)\,, \\ (1 - \gamma_5 \not{n}_p)\,(\pm \not{p} + m) &= (1 \mp \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}})\,(\pm \not{p} + m)\,;\end{aligned}$$

or alternatively,

$$\begin{aligned}P(n_p)\,\Lambda_\pm(p) &= \tfrac{1}{2}\,(1 \pm \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}})\,\Lambda_\pm(p)\,, \\ P(-n_p)\,\Lambda_\pm(p) &= \tfrac{1}{2}\,(1 \mp \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}})\,\Lambda_\pm(p)\,.\end{aligned} \tag{3.74}$$

Just as expected, $P(n_p)$ projects out the positive-helicity component from a positive-energy state and the negative-helicity component from a negative-energy state, and similarly, $P(-n_p)$ projects out the negative-helicity component from a positive-energy state and the positive-helicity component from a negative-energy state.

Projection operators are very useful in practice. In expressions where specific states are selectively considered, they make possible the use of closure relations and unnecessary explicit calculations of the spinors, replacing these by known spin matrices. For example, the probability of a certain process may be given by $\sum_i (\bar{u}_f \Gamma u_i)(\bar{u}_i \Gamma' u_f)$, where the summation is to be performed over the two positive-energy spinors u_i. Then the sum $\sum_i u_i \bar{u}_i$ can be replaced by $\not{p} + m = 2m\,\Lambda_+(p)$. On the other hand, if instead of summing over both spin states, one calculates rather the probability for some given polarization, then one just inserts the operator $P(n_p)$ to project out the appropriate spin component.

3.4 The Lagrangian for a Free Dirac Particle

As we have seen in the last chapter, a Lagrangian completely defines the dynamics of any given system and embodies all of its symmetries. The Lagrangian for a free Dirac field is

$$\mathcal{L} = \overline{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi\,. \tag{3.75}$$

Since $\mathcal{L}$ has dimension $[E]^4$, the Dirac field must have dimension $[E]^{3/2}$. The Euler–Lagrange equations for the field variables ψ and $\overline{\psi}$,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\overline{\psi}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} &= (\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0\,,\\ \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} &= -(\partial_\mu\overline{\psi}\,\mathrm{i}\gamma^\mu + m\overline{\psi}) = 0\,,\end{aligned}$$

correctly reproduce the Dirac equations (10) and (11). Noether's theorem introduced in the last chapter will now be applied to derive the conserved currents associated with the symmetries of the system.

It is clear that the Dirac Lagrangian (75) is invariant to any constant translation,

$$\begin{aligned} x^\mu \to \quad & x'^\mu = x^\mu - a^\mu\,,\\ \psi(x) \to & \,\psi'(x') = \psi(x)\,.\end{aligned}$$

The associated current is the energy-momentum tensor

$$\begin{aligned}\mathcal{T}^\mu{}_\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\partial_\nu\psi + \partial_\nu\overline{\psi}\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \delta^\mu{}_\nu\,\mathcal{L}\\ &= \overline{\psi}\,\mathrm{i}\gamma^\mu\partial_\nu\psi\,,\end{aligned} \tag{3.76}$$

where ψ is a solution to the Dirac equation and, therefore, $\mathcal{L} = 0$. Since the current is conserved, $\partial_\mu\mathcal{T}^\mu{}_\nu = 0$, the corresponding 'charge' or momentum

$$P_\nu = \int \mathrm{d}^3x\,\mathcal{T}^0{}_\nu = \int \mathrm{d}^3x\,\overline{\psi}\,\mathrm{i}\gamma^0\,\partial_\nu\psi \tag{3.77}$$

is a constant of the motion. In particular, its zero-component defines the energy or Hamiltonian of the system

$$\begin{aligned} H = P_0 &= \int \mathrm{d}^3x\,\overline{\psi}\,\mathrm{i}\gamma^0\,\partial_0\psi\\ &= \int \mathrm{d}^3x\,\psi^\dagger\gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\nabla + m)\psi\,,\end{aligned} \tag{3.78}$$

where we can recognize the Hamiltonian operator $\hat{H} = \gamma^0(-\mathrm{i}\boldsymbol{\gamma}\cdot\nabla + m)$.

From the Lorentz transformation properties of the bilinear covariants $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\psi$ (see Table 3.1), $\mathcal{L}$ is seen to be Lorentz-invariant. In an infinitesimal Lorentz transformation, parameterized by $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$,

$$\begin{aligned} x^\mu \to\ & x'^\mu \;\; = x^\mu + \epsilon^\mu{}_\nu\, x^\nu\,,\\ \psi(x) \to\ & \psi'(x') = S(1+\epsilon)\,\psi(x)\,,\end{aligned}$$

the field variation is, according to (28),

$$\delta_0\psi(x) = -\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu} J_{\mu\nu}\,\psi(x)\,, \tag{3.79}$$

where $J_{\mu\nu}$ are the generators of the infinitesimal Lorentz transformation,

$$J_{\mu\nu} = L_{\mu\nu} + \tfrac{1}{2}\,\sigma_{\mu\nu}, \tag{3.80}$$

with $L_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)$ representing the orbital part and $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$ the intrinsic part. To evaluate the Noether current density associated with this symmetry, the general expression (2.189) becomes in this case

$$\mathcal{M}^\mu_{\rho\sigma} \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\frac{\delta_0\psi}{\delta\epsilon^{\rho\sigma}} + \frac{\delta_0\overline{\psi}}{\delta\epsilon^{\rho\sigma}}\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} + \mathcal{L}\,\frac{\delta x^\mu}{\delta\epsilon^{\rho\sigma}}\,. \tag{3.81}$$

As only the first term on the right-hand side is nonvanishing, using

$$\frac{\delta_0\psi}{\delta\epsilon^{\rho\sigma}} = -\mathrm{i}J_{\rho\sigma}\psi \;=\; (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\psi - \frac{\mathrm{i}}{2}\sigma_{\rho\sigma}\psi \tag{3.82}$$

immediately carries (81) into the desired result

$$\mathcal{M}^\mu_{\rho\sigma} \;=\; x_\rho\mathcal{T}^\mu{}_\sigma - x_\sigma\mathcal{T}^\mu{}_\rho + \frac{1}{2}\overline{\psi}\,\gamma^\mu\sigma_{\rho\sigma}\,\psi\,. \tag{3.83}$$

The associated conserved ‘charge’ is the angular momentum tensor

$$\begin{aligned} M_{\rho\sigma} &= \int \mathrm{d}^3x\,\mathcal{M}^0_{\rho\sigma} \\ &= \int \mathrm{d}^3x\,(\,x_\rho\mathcal{T}^0{}_\sigma - x_\sigma\mathcal{T}^0{}_\rho) + \frac{1}{2}\int \mathrm{d}^3x\,\overline{\psi}\,\gamma^0\sigma_{\rho\sigma}\,\psi\,, \end{aligned} \tag{3.84}$$

where evidently the first integral on the last line represents the orbital component, and the second, the intrinsic component.

The Lagrangian (75) is also invariant to complex phase transformations of the fields,

$$\begin{aligned} \psi(x) &\to \psi'(x) = \mathrm{e}^{-\mathrm{i}\alpha}\,\psi(x) \approx \psi(x) - \mathrm{i}\alpha\,\psi(x), \\ \overline{\psi}(x) &\to \overline{\psi}'(x) = \mathrm{e}^{\mathrm{i}\alpha}\,\overline{\psi}(x) \;\approx \overline{\psi}(x) + \mathrm{i}\alpha\,\overline{\psi}(x), \end{aligned} \tag{3.85}$$

where α stands for a real constant. These internal transformations change only the fields, leaving untouched their coordinate arguments,

$$\frac{\delta_0\psi}{\delta\alpha} = -\mathrm{i}\psi\,, \qquad \frac{\delta_0\overline{\psi}}{\delta\alpha} = \;\mathrm{i}\overline{\psi}\,. \tag{3.86}$$

The associated conserved current density and charge are

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\frac{\delta_0\psi}{\delta\alpha} + \frac{\delta_0\overline{\psi}}{\delta\alpha}\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} \;=\; \overline{\psi}\gamma^\mu\psi\,, \tag{3.87}$$

$$Q = \int \mathrm{d}^3x\,j^0 \;=\; \int \mathrm{d}^3x\,\psi^\dagger\psi\,. \tag{3.88}$$

3.5 Quantization of the Dirac Field

In Dirac's theory, the probability density given by the time component of a conserved current, $j^0 = \psi^\dagger\psi$ for $\psi = \psi_+$ or $\psi = \psi_-$, is evidently positive-definite. This result, by avoiding one of the obstacles initially met by the Klein–Gordon equation, makes it possible to interpret the Dirac equation as the basic equation for a *one-particle* system. However, Dirac could not prevent the presence of negative-energy solutions, and it is a measure of his genius to be able to turn this apparent difficulty to his advantage, giving us at the same time the novel concept of *antiparticle*. As we have seen in the last chapter, the negative-energy solution to the Klein–Gordon equation can be interpreted as the wave function of an antiboson of electric charge opposite to that of the particle described by the positive-energy solution, and the current density of the theory must be considered not as a probability current density but rather as a charge current density. A similar interpretation applies to the present case as well, and should even emerge quite naturally when one considers processes in which particles are created or destroyed, such as $\mathrm{n} \to \mathrm{p} + \mathrm{e}^- + \bar\nu$ or $\gamma \to \mathrm{e}^+ + \mathrm{e}^-$, because, clearly, what is conserved then is not the probability of finding a given particle in the space volume but rather the total electric charge of the system. The problem becomes a *many-body* problem for which the quantum field theory is the most appropriate approach. The classical Dirac field is then treated as a field operator which describes the creation and annihilation of fermions and antifermions at all points in space-time, paralleling the role played by the quantized Klein–Gordon field for bosons and antibosons. However, there is a fundamental difference between the two cases that must be taken into account in the formulation, namely, the existence for fermions of a rule (the Pauli exclusion principle) that forbids the presence of more than one fermion of the same kind in the same state.

The four solutions to the Dirac equation for a free particle

$$\begin{aligned}
\psi^{(+)}_{\boldsymbol{p},s}(x) &= C_{\boldsymbol{p}}\psi_+(x) \;=\; C_{\boldsymbol{p}} u(p,s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,, \\
\psi^{(-)}_{-\boldsymbol{p},-s}(x) &= C_{\boldsymbol{p}}\psi_-(x) \;=\; C_{\boldsymbol{p}} v(p,s)\,\mathrm{e}^{\mathrm{i}p\cdot x}\,,
\end{aligned} \tag{3.89}$$

are the eigenvectors of $\hat H$ with energies E_p and $-E_p$ ($p_0 = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$), and of spin $s_z = \pm 1/2$. According to (66) they form a complete set in the spinor representation and, with normalization $C_{\boldsymbol{p}} = 1/\sqrt{(2\pi)^3\, 2E_{\boldsymbol{p}}}$, satisfy the orthonormality relations

$$\begin{aligned}
\int \mathrm{d}^3x\, \psi^{(+)\dagger}_{\boldsymbol{p}',s'}(x)\, \psi^{(+)}_{\boldsymbol{p},s}(x) &= \delta(\boldsymbol{p}' - \boldsymbol{p})\,\delta_{s's}\,, \\
\int \mathrm{d}^3x\, \psi^{(-)\dagger}_{\boldsymbol{p}',s'}(x)\, \psi^{(-)}_{\boldsymbol{p},s}(x) &= \delta(\boldsymbol{p}' - \boldsymbol{p})\,\delta_{s's}\,, \\
\int \mathrm{d}^3x\, \psi^{(+)\dagger}_{\boldsymbol{p}',s'}(x)\, \psi^{(-)}_{-\boldsymbol{p},-s}(x) = \int \mathrm{d}^3x\, \psi^{(-)\dagger}_{-\boldsymbol{p}',-s'}(x)\, \psi^{(+)}_{\boldsymbol{p},s}(x) &= 0\,.
\end{aligned} \tag{3.90}$$

The wave function $\psi^{(+)}_{\boldsymbol{p},s}(x)$ is the solution for a positive-energy state of momentum $\boldsymbol{p}$ and polarization s, whereas $\psi^{(-)}_{-\boldsymbol{p},-s}(x)$ is the wave function for a negative-energy state of momentum $-\boldsymbol{p}$ and polarization $-s$, which is however more conveniently reinterpreted as describing a state of an antiparticle of positive energy, momentum $\boldsymbol{p}$, and polarization s. Note that, apart from the presence of the spin, the situation is exactly the same as in the case of the solutions to the Klein–Gordon equation and therefore, this reinterpretation can be similarly justified.

The formalism of the classical fields discussed in the previous section is converted into a quantum field theory by simply treating the Dirac field ψ as a quantum operator. The fields ψ and $\overline{\psi}$ are expanded over the complete set of eigenspinors $\psi^{(\pm)}_{p,s}$:

$$\begin{aligned}\psi(x) &= \sum_{\boldsymbol{p},s} \left[\psi^{(+)}_{\boldsymbol{p},s}(x)\, b(\boldsymbol{p},s) + \psi^{(-)}_{-\boldsymbol{p},-s}(x)\, d^\dagger(\boldsymbol{p},s)\right] , \\ \overline{\psi}(x) &= \sum_{\boldsymbol{p},s} \left[\overline{\psi}^{(+)}_{\boldsymbol{p},s}(x)\, b^\dagger(\boldsymbol{p},s) + \overline{\psi}^{(-)}_{-\boldsymbol{p},-s}(x)\, d(\boldsymbol{p},s)\right] \end{aligned} \tag{3.91}$$

(where $\sum_{\boldsymbol{p}} = \int \mathrm{d}^3p$), and the expansion coefficients are treated as operators of creation and destruction:

$b(\boldsymbol{p},s)$ destroys a particle of momentum $\boldsymbol{p}$ and polarization s,
$d(\boldsymbol{p},s)$ destroys an antiparticle of momentum $\boldsymbol{p}$ and polarization s,
$b^\dagger(\boldsymbol{p},s)$ creates a particle of momentum $\boldsymbol{p}$ and polarization s,
$d^\dagger(\boldsymbol{p},s)$ creates an antiparticle of momentum $\boldsymbol{p}$ and polarization s.

3.5.1 Spins and Statistics

The creation and annihilation operators applied on the ground state, the vacuum $|0\rangle$, produce states of one or several particles called the Fock states. Thus, for example, the state of one particle of momentum p (suppressing spin for the moment) is given by

$$|\boldsymbol{p}\rangle = C^{-1}_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}}\, |0\rangle , \tag{3.92}$$

and the state of two identical particles of momenta p and p' is given by

$$|\boldsymbol{p},\boldsymbol{p}'\rangle = C^{-1}_{\boldsymbol{p}'} C^{-1}_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}'} b^\dagger_{\boldsymbol{p}}\, |0\rangle\ . \tag{3.93}$$

The probability for finding two particles of the same kind of momenta p and p' in an arbitrary physical state Ψ is $|\langle \Psi\,|\,\boldsymbol{p},\boldsymbol{p}'\rangle|^2$. As the two particles are identical, they cannot be distinguished by any experiment; all that can be said is that one of them has momentum p and the other momentum p', a statement that can be translated into the equation

$$|\langle \Psi\,|\,\boldsymbol{p},\boldsymbol{p}'\rangle|^2 = |\langle \Psi\,|\,\boldsymbol{p}',\boldsymbol{p}\rangle|^2 , \tag{3.94}$$

which implies (for some real ϕ)

$$\langle \Psi \,|\, \boldsymbol{p}, \boldsymbol{p}' \,\rangle \;=\; \mathrm{e}^{i\phi} \,\langle \Psi \,|\, \boldsymbol{p}', \boldsymbol{p} \rangle \;.$$

Since two successive permutations of the particles must return $|\boldsymbol{p}, \boldsymbol{p}'\rangle$ to the same state, this also means

$$\langle \Psi \,|\, \boldsymbol{p}, \boldsymbol{p}' \,\rangle \;=\; \pm \,\langle \Psi \,|\, \boldsymbol{p}', \boldsymbol{p} \rangle \;, \tag{3.95}$$

or, since Ψ is arbitrary,

$$|\boldsymbol{p}, \boldsymbol{p}'\rangle \;=\; \pm\; |\boldsymbol{p}', \boldsymbol{p}\rangle \;. \tag{3.96}$$

The two solutions correspond to the two possible statistics for identical quantum particles: in the Bose–Einstein statistics the Fock states are *symmetric* under a permutation of any two particles, while in the Fermi–Dirac statistics they are *antisymmetric*. Particles obeying the Bose–Einstein statistics are referred to as bosons, and those obeying the Fermi–Dirac statistics as fermions. The creation and annihilation operators for a boson field satisfy *commutation relations*, whereas those for a fermion field satisfy *anticommutation relations.*

Let us assume from now on that b and $b^\dagger$ are operators for a fermion field. The anticommutation relation

$$b_{\boldsymbol{p}'} b_{\boldsymbol{p}} + b_{\boldsymbol{p}} b_{\boldsymbol{p}'} \equiv \{b_{\boldsymbol{p}'}, b_{\boldsymbol{p}}\} = 0$$

implies that $b_{\boldsymbol{p}} b_{\boldsymbol{p}} = 0$, or that two identical fermions cannot occupy the same state. The operator for the occupation number of the individual state p is

$$\mathcal{N}_b = b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} \,.$$

It follows from the anticommutation rules that $\mathcal{N}_b(1-\mathcal{N}_b) = 0$, which means that the number of fermions of a given kind occupying a given individual state is either 0 or 1 . This is the *Pauli exclusion principle.*

There exists in quantum field theory a general theorem giving a connection between spins and statistics. It states that for a Lorentz-invariant local field theory in four-dimensional space-time admitting a unique vacuum state, the fields of integral spins are quantized as Bose–Einstein fields and the fields of half-integral spins are quantized as Fermi–Dirac fields if the microcausality condition is satisfied. A local theory means the Lagrangian density describing the theory contains fields that refer to a single space-time point. The microcausality condition means the local density operators do not interfere, that is, they commute (or anticommute) for spacelike separations. The predictions of this fundamental theorem are in perfect agreement with experimental observations.

3.5.2 Dirac Field Observables

From the above arguments, the Dirac field operator ψ and its canonical momentum, $\mathrm{i}\psi^\dagger$, must satisfy the following *anticommutation* (rather than commutation) quantization rules:

$$\begin{aligned}&\{\psi_a(t,\boldsymbol{x}),\,\psi_b^\dagger(t,\boldsymbol{y})\}=\delta_{ab}\delta(\boldsymbol{x}-\boldsymbol{y})\,;\\&\{\psi_a(t,\boldsymbol{x}),\,\psi_b(t,\boldsymbol{y})\}=0\,;\end{aligned}\tag{3.97}$$

which lead to the corresponding algebra for the associated creation and annihilation operators:

$$\begin{aligned}\{b(\boldsymbol{p}',s'),\,b^\dagger(\boldsymbol{p},s)\}&=\delta_{ss'}\delta(\boldsymbol{p}'-\boldsymbol{p})\,;\\\{d(\boldsymbol{p}',s'),\,d^\dagger(\boldsymbol{p},s)\}&=\delta_{ss'}\delta(\boldsymbol{p}'-\boldsymbol{p})\,;\\\{b(\boldsymbol{p}',s'),\,b(\boldsymbol{p},s)\}&=\{b(\boldsymbol{p}',s'),\,d(\boldsymbol{p},s)\}=0\,;\\\{d(\boldsymbol{p}',s'),\,d(\boldsymbol{p},s)\}&=\{b(\boldsymbol{p}',s'),\,d^\dagger(\boldsymbol{p},s)\}=0\,.\end{aligned}\tag{3.98}$$

The Hamiltonian H can be expressed in terms of the static operators by substituting (91) in (78) and using (89):

$$\begin{aligned}H&=\int\mathrm{d}^3x\,\psi^\dagger(x)\sum_{\boldsymbol{p},s}E_{\boldsymbol{p}}\left[\psi^{(+)}_{\boldsymbol{p},s}(x)\,b(\boldsymbol{p},s)-\psi^{(-)}_{-\boldsymbol{p},-s}(x)\,d^\dagger(\boldsymbol{p},s)\right]\\&=\sum_{\boldsymbol{p},s}E_{\boldsymbol{p}}\left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s)-d(\boldsymbol{p},s)\,d^\dagger(\boldsymbol{p},s)\right]\,.\end{aligned}\tag{3.99}$$

If ψ and $\overline{\psi}$ were classical fields, b and d would be c-number coefficients, the second term in (99) would be negative, and the field energy H could not be positive-definite. Therefore a classical Dirac field cannot exist. On the other hand, if b and d are operators that commute as in the case of the boson fields, the energy again will not be positive definite and will not have a lower bound. The only possible way to have a positive value for the second term in (99) is to make $d(\boldsymbol{p},s)d^\dagger(\boldsymbol{p},s)$ change signs when d and $d^\dagger$ are interchanged, that is, to require that d and $d^\dagger$ (and by extension b and $b^\dagger$) obey the anticommutation relations (98). The Hamiltonian operator is then given by

$$H=\sum_{\boldsymbol{p},s}E_{\boldsymbol{p}}\,[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s)+d^\dagger(\boldsymbol{p},s)\,d(\boldsymbol{p},s)\,]\,.\tag{3.100}$$

As in the boson field case, an additive constant, interpreted as the vacuum energy, has been dropped. The total energy of the field appears then as a sum of positive-energy contributions from all different modes of fermions and antifermions.

The procedure leading to (100) can be summarized by the formula

$$\begin{aligned}H&=\int\mathrm{d}^3x\;:\,\overline{\psi}\mathrm{i}\gamma^0\partial_0\psi\,:\\&=\sum_{\boldsymbol{p},s}E_{\boldsymbol{p}}\left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s)+d^\dagger(\boldsymbol{p},s)\,d(\boldsymbol{p},s)\right]\,.\end{aligned}\tag{3.101}$$

It consists in writing the products of the creation and annihilation operators in the *normal order* (symbolized by : :) by reordering the factors such that the creation operators are to the left of the destruction operators taking into account all sign changes arising from permutations of operators in accordance with their statistics. The final additive constant term, independent of operators, which results from these operations, is identified with the vacuum expectation value $\langle 0 \,|\, H \,|\, 0\rangle$ and dropped. In what follows, the normal order of field products in the expressions for observables is always assumed, even though the notation : : may not be used explicitly.

The field momentum (77) can be similarly calculated by noting that $\nabla\psi^{(+)}_{\boldsymbol{p},s}(x) = \mathrm{i}\boldsymbol{p}\,\psi^{(+)}_{\boldsymbol{p},s}(x)$ and $\nabla\psi^{(-)}_{-\boldsymbol{p},-s}(x) = -\mathrm{i}\boldsymbol{p}\,\psi^{(-)}_{-\boldsymbol{p},-s}(x)$ and using the orthogonality properties of the basis functions:

$$
\begin{aligned}
\boldsymbol{P} &= -\mathrm{i}\int \mathrm{d}^3x \;:\psi^\dagger(x)\,\nabla\psi(x): \\
&= \sum_{\boldsymbol{p},s} \boldsymbol{p} : \left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s) - d(\boldsymbol{p},s)\,d^\dagger(\boldsymbol{p},s)\right] : \\
&= \sum_{\boldsymbol{p},s} \boldsymbol{p} \left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s) + d^\dagger(\boldsymbol{p},s)\,d(\boldsymbol{p},s)\right] . \qquad (3.102)
\end{aligned}
$$

Finally, the charge operator (88) is found to be

$$
\begin{aligned}
Q &= \int \mathrm{d}^3x \;:\psi^\dagger(x)\,\psi(x): \\
&= \sum_{\boldsymbol{p},s} : \left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s) + d(\boldsymbol{p},s)\,d^\dagger(\boldsymbol{p},s)\right] : \\
&= \sum_{\boldsymbol{p},s} \left[b^\dagger(\boldsymbol{p},s)\,b(\boldsymbol{p},s) - d^\dagger(\boldsymbol{p},s)\,d(\boldsymbol{p},s)\right] . \qquad (3.103)
\end{aligned}
$$

3.5.3 Fock Space

To gain a better physical understanding of the formalism, let us study the observables for states in the Fock space and in particular for the one-fermion or one-antifermion states

$$
|p,s\rangle = C_{\boldsymbol{p}}^{-1} b^\dagger(\boldsymbol{p},s)\;|0\rangle \;, \qquad |\overline{p,s}\rangle = C_{\boldsymbol{p}}^{-1} d^\dagger(\boldsymbol{p},s)\;|0\rangle \;. \qquad (3.104)
$$

First note the identity valid for any three arbitrary operators

$$
[AB, C] = A\,\{B, C\} - \{A, C\}\,B\,. \qquad (3.105)
$$

The operator algebra (98) and the above expressions for H, $\boldsymbol{P}$, and Q can be used to derive the following relations:

$$
\begin{aligned}
&[H,\, b^\dagger(\boldsymbol{p},s)] = E_{\boldsymbol{p}} b^\dagger(\boldsymbol{p},s)\,, &\quad &[H,\, d^\dagger(\boldsymbol{p},s)] = E_{\boldsymbol{p}} d^\dagger(\boldsymbol{p},s)\,,\\
&[\boldsymbol{P},\, b^\dagger(\boldsymbol{p},s)] = \boldsymbol{p}\, b^\dagger(\boldsymbol{p},s)\,, &\quad &[\boldsymbol{P},\, d^\dagger(\boldsymbol{p},s)] = \boldsymbol{p}\, d^\dagger(\boldsymbol{p},s)\,,\\
&[Q,\, b^\dagger(\boldsymbol{p},s)] = b^\dagger(\boldsymbol{p},s)\,, &\quad &[Q,\, d^\dagger(\boldsymbol{p},s)] = -d^\dagger(\boldsymbol{p},s)\,.
\end{aligned}
$$

By taking their Hermitian conjugates while recalling the hermiticity of the operators H, $\boldsymbol{P}$, and Q, one obtains similar equations involving $b(\boldsymbol{p},s)$ or $d(\boldsymbol{p},s)$ in place of $b^\dagger(\boldsymbol{p},s)$ or $d^\dagger(\boldsymbol{p},s)$. Then the energies, momenta, and charges for one-particle states are given by

$$
\begin{aligned}
H\;|p,s\rangle &= [H,\, b^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = E_{\boldsymbol{p}}\;|p,s\rangle\;,\\
\boldsymbol{P}\;|p,s\rangle &= [\boldsymbol{P},\, b^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = \;\boldsymbol{p}\;\;|p,s\rangle\;,\\
Q\;|p,s\rangle &= [Q,\, b^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = \;\;|p,s\rangle\;,
\end{aligned}
$$

and for one-antiparticle states by

$$
\begin{aligned}
H\;|\overline{p,s}\rangle &= [H,\, d^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = E_{\boldsymbol{p}}\;|\overline{p,s}\rangle\;,\\
\boldsymbol{P}\;|\overline{p,s}\rangle &= [\boldsymbol{P},\, d^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = \;\boldsymbol{p}\;\;|\overline{p,s}\rangle\;,\\
Q\;|\overline{p,s}\rangle &= [Q,\, d^\dagger(\boldsymbol{p},s)]\;|0\rangle\; C_{\boldsymbol{p}}^{-1} = -\;|\overline{p,s}\rangle\;.
\end{aligned}
$$

It is now clear that $b^\dagger(\boldsymbol{p},s)$ increases the energy of the system by $E_{\boldsymbol{p}}$, its momentum by $\boldsymbol{p}$, and its charge by a unit of charge, while $d^\dagger(\boldsymbol{p},s)$ also increases the energy of the system by $E_{\boldsymbol{p}}$, its momentum by $\boldsymbol{p}$, but reduces its charge by a unit of charge. One can similarly show that $b(\boldsymbol{p},s)$ and $d(\boldsymbol{p},s)$ both reduce the energy and momentum of the system by $E_{\boldsymbol{p}}$ and $\boldsymbol{p}$, but while $b(\boldsymbol{p},s)$ reduces the charge by one unit, $d(\boldsymbol{p},s)$ increases it by the same amount. In other words, $b^\dagger(\boldsymbol{p},s)$ creates and $b(\boldsymbol{p},s)$ destroys a particle of energy $E_{\boldsymbol{p}}$, momentum $\boldsymbol{p}$, and of unit charge, whereas $d^\dagger(\boldsymbol{p},s)$ creates and $d(\boldsymbol{p},s)$ destroys an antiparticle of energy $E_{\boldsymbol{p}}$, momentum $\boldsymbol{p}$, and of charge equal in magnitude but opposite in sign to the unit charge. Since $E_{\boldsymbol{p}}^2-\boldsymbol{p}^2=m^2$ in both cases, a particle and its conjugate antiparticle that are associated with the same field operator have equal masses.

The polarization states can be understood as follows. The intrinsic part of the angular momentum tensor, given by (84)

$$
S_{ij} = \tfrac{1}{2}\int \mathrm{d}^3x\, \psi^\dagger \sigma_{ij}\psi\,, \tag{3.106}
$$

leads to the definition for the spin operator

$$
S^i = \mathrm{i}\epsilon^{ijk} S_{jk}\,. \tag{3.107}
$$

To simplify we consider just its third component

$$
\begin{aligned}
S^3 \equiv S_z = \sum_{\boldsymbol{p},s,s'} (2\pi)^3 C_{\boldsymbol{p}}^2 \Big[& u^\dagger(\boldsymbol{p},s')\tfrac{1}{2}\,\Sigma_z u(\boldsymbol{p},s)\, b^\dagger(\boldsymbol{p},s')b(\boldsymbol{p},s)\\
&- v^\dagger(\boldsymbol{p},s')\tfrac{1}{2}\,\Sigma_z v(\boldsymbol{p},s)\, d^\dagger(\boldsymbol{p},s)d(\boldsymbol{p},s')\\
&+ u^\dagger(\boldsymbol{p},s')\tfrac{1}{2}\,\Sigma_z v(-\boldsymbol{p},s)\, b^\dagger(\boldsymbol{p},s')d^\dagger(-\boldsymbol{p},s)\,\mathrm{e}^{2\mathrm{i}Et}\\
&+ v^\dagger(-\boldsymbol{p},s)\tfrac{1}{2}\,\Sigma_z u(\boldsymbol{p},s')\, d(-\boldsymbol{p},s)b(\boldsymbol{p},s')\,\mathrm{e}^{-2\mathrm{i}Et} \Big]\,.
\end{aligned}
$$

By definition $b(\boldsymbol{p})\ |0\rangle = d(\boldsymbol{p})\ |0\rangle = 0$, and by rotational invariance $S_z\ |0\rangle = 0$. Application of S_z on a one-particle state and a one-antiparticle state yields

$$\begin{aligned} S_z b^\dagger(\boldsymbol{k}, r)\ |0\rangle &= \left[S_z, b^\dagger(\boldsymbol{k}, r)\right]\ |0\rangle \\ &= +(2\pi)^3 C_{\boldsymbol{k}}^2 \sum_s u^\dagger(\boldsymbol{k}, s) \tfrac{1}{2}\, \Sigma_z u(\boldsymbol{k}, r) b^\dagger(\boldsymbol{k}, s)\ |0\rangle\ , \end{aligned} \tag{3.108}$$

$$\begin{aligned} S_z d^\dagger(\boldsymbol{k}, r)\ |0\rangle &= \left[S_z, d^\dagger(\boldsymbol{k}, r)\right]\ |0\rangle \\ &= -(2\pi)^3 C_{\boldsymbol{k}}^2 \sum_s v^\dagger(\boldsymbol{k}, r) \tfrac{1}{2}\, \Sigma_z v(\boldsymbol{k}, s) d^\dagger(\boldsymbol{k}, s)\ |0\rangle\ . \end{aligned} \tag{3.109}$$

If the z axis is chosen in the same direction as the momentum vector $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$, then $\frac{1}{2}\,\Sigma_z = \frac{1}{2}\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{k}}$ represents the helicity. In the rest frame, where $\boldsymbol{k} = 0$, it is then convenient to choose the spinors $u(0, r)$ and $v(0, r)$ as eigenspinors of $\frac{1}{2}\,\Sigma_z$ with eigenvalues $+1/2$ and $-1/2$ for $r = 1, 2$. In a general frame where $\boldsymbol{k} = |\boldsymbol{k}|\hat{z} \neq 0$, the spinors $u(\boldsymbol{k}, r)$ and $v(\boldsymbol{k}, r)$ remain eigenspinors of $\frac{1}{2}\,\Sigma_z$ with eigenvalues λ_r ($\lambda_1 = +1/2$, $\lambda_2 = -1/2$):

$$\tfrac{1}{2}\,\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{k}}\, u(\boldsymbol{k}, r) = \lambda_r u(\boldsymbol{k}, r)\,, \qquad \tfrac{1}{2}\,\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{k}}\, v(\boldsymbol{k}, r) = \lambda_r v(\boldsymbol{k}, r)\,.$$

Using the normalization (51) and (52), we immediately obtain

$$S_z b^\dagger(\boldsymbol{k}, r)\ |0\rangle = \ \lambda_r\, b^\dagger(\boldsymbol{k}, r)\ |0\rangle\ , \tag{3.110}$$

$$S_z d^\dagger(\boldsymbol{k}, r)\ |0\rangle = -\lambda_r\, d^\dagger(\boldsymbol{k}, r)\ |0\rangle\ . \tag{3.111}$$

These results tell us that $b^\dagger(\boldsymbol{k}, 1)$ and $d^\dagger(\boldsymbol{k}, 2)$ create states of helicity $+1/2$, while $b^\dagger(\boldsymbol{k}, 2)$ and $d^\dagger(\boldsymbol{k}, 1)$ create states of helicity $-1/2$.

Just as for the boson fields, the quantum fields $\psi(x)$ and $\overline{\psi}(x)$ are related to the c-valued wave functions $\psi^{(+)}_{\boldsymbol{p},s}(x)$ and $\psi^{(-)}_{-\boldsymbol{p},-s}(x)$. For example, $\psi^{(+)}_{\boldsymbol{p},s}(x)$ can be interpreted as the annihilation amplitude of a particle at point x, $\langle 0\,|\,\psi(x)\,|\,ps\rangle$, and the spinor $u(p, s)$ is associated with an incoming fermion of momentum $\boldsymbol{p}$ and polarization s. Similarly, $\psi^{(-)}_{-\boldsymbol{p},-s}(x)$ represents the creation amplitude of an antiparticle at x, and the spinor $v(p, s)$ is associated with an outgoing antifermion of momentum p and polarization s. To summarize,

$\langle 0\|\psi(x)\|ps\rangle =$	$u(p, s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}$	$\rightarrow\!\!\bullet$	annihilation of a particle,
$\langle ps\|\overline{\psi}(x)\|0\rangle =$	$\bar{u}(p, s)\,\mathrm{e}^{\mathrm{i}p\cdot x}$	$\bullet\!\!\rightarrow$	creation of a particle,
$\langle\overline{ps}\|\psi(x)\|0\rangle =$	$v(p, s)\,\mathrm{e}^{\mathrm{i}p\cdot x}$	$\bullet\!\!\leftarrow$	creation of an antiparticle,
$\langle 0\|\overline{\psi}(x)\|\overline{ps}\rangle =$	$\bar{v}(p, s)\,\mathrm{e}^{-\mathrm{i}p\cdot x}$	$\leftarrow\!\!\bullet$	annihilation of an antiparticle.

3.6 Zero-Mass Fermions

When the field is massless, the Dirac theory may be formulated in terms of two-component spinors. This simplification proves to be quite useful in the study of neutrinos, which have very small masses, or of particles of non-vanishing masses at very high energies, where their masses can be neglected in comparison with their kinetic energies.

If ψ is a solution to the Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0\,, \tag{3.112}$$

$\gamma_5\psi$ obeys the equation

$$(\mathrm{i}\gamma^\mu \partial_\mu + m)\gamma_5\psi = 0\,. \tag{3.113}$$

The two equations are different for a nonvanishing mass. But when $m = 0$ they become identical and the spinors ψ and $\gamma_5\psi$ are proportional to each other. In other words, for $m = 0$ the Dirac equation may be written as

$$\mathrm{i}\frac{\partial}{\partial t}\psi = \hat{H}\psi\,, \tag{3.114}$$

where $\hat{H} = -\mathrm{i}\gamma_0\boldsymbol{\gamma}\cdot\nabla$. Since $[\hat{H}, \gamma_5] = 0$, the matrices $\hat{H}$ and γ_5 are simultaneously diagonalizable, and common eigenfunctions can be found for $\hat{H}$ and γ_5 . As $\gamma_5^2 = 1$, the eigenvalues of γ_5 are ± 1 . An eigenspinor with eigenvalue $+1$ for γ^5 is said to have a *positive chirality*; when its eigenvalue for γ^5 is -1, it is said to have a *negative chirality*. We now proceed to describe these spinors.

For $m = 0$, (54) reduces to

$$\begin{aligned} \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\,u(\boldsymbol{p},h) &= \;\;E\,u(\boldsymbol{p},h)\,,\\ \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\,v(-\boldsymbol{p},h) &= -E\,v(-\boldsymbol{p},h)\,, \end{aligned} \tag{3.115}$$

where $E = |\boldsymbol{p}|$. As already mentioned, $u(\boldsymbol{p}, h)$ and $v(\boldsymbol{p}, h)$ can always be chosen as eigenspinors of the helicity operator:

$$\begin{aligned} \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}\,u(\boldsymbol{p},h) &= 2h\,u(\boldsymbol{p},h)\,,\\ \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}\,v(-\boldsymbol{p},h) &= 2h\,v(-\boldsymbol{p},h)\,. \end{aligned} \tag{3.116}$$

From the definition $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, one gets $\gamma_5\gamma^0 = -\mathrm{i}\,\gamma^1\gamma^2\gamma^3$, and so $\gamma_5\gamma^0\gamma^1 = \mathrm{i}\,\gamma^2\gamma^3 = \sigma^{23} = \Sigma^1$. In general, $\gamma_5\gamma^0\gamma^i = \Sigma^i$. Applying γ_5 from the left on both sides of (115) and using (116), one gets

$$\begin{aligned} \gamma_5\,u(\boldsymbol{p},h) &= \;\;\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}\,u(\boldsymbol{p},h) = \;\;2h\,u(\boldsymbol{p},h)\,,\\ \gamma_5\,v(-\boldsymbol{p},h) &= -\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}\,v(-\boldsymbol{p},h) = -2h\,v(-\boldsymbol{p},h)\,. \end{aligned} \tag{3.117}$$

This means in particular that for an arbitrary vector $\boldsymbol{p}$, the spinors $u(\boldsymbol{p}, +1/2)$ and $v(\boldsymbol{p}, -1/2)$ are eigenspinors of γ^5 of positive chirality $+1$, while $u(\boldsymbol{p}, -1/2)$ and $v(\boldsymbol{p}, +1/2)$ have negative chirality -1 . Thus, for a zero-mass particle, chirality and helicity are equivalent and are Lorentz-invariant. On the other hand, for a particle with a nonvanishing mass, chirality is not well defined, but states of such a particle can still be identified by their helicities. As it is the scalar product of two three-vectors, $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$, the helicity is invariant to spatial rotations, which makes a description of states in terms of helicities often useful. However, a general Lorentz transformation will mix up states of different helicities. For example, a particle of nonvanishing mass with a positive helicity in a given inertial frame will have a negative helicity in a frame in which its direction of motion is reversed. Thus, a Dirac particle with a nonzero mass must occur in both helicity states.

Any spinor ψ, massive or not, can be decomposed into two components of well-defined chiralities, called the *Weyl spinors*,

$$\begin{aligned} \psi &= \psi_{\mathrm{R}} + \psi_{\mathrm{L}}\,, \\ \psi_{\mathrm{R}} &= \tfrac{1}{2}\,(1+\gamma_5)\psi, \quad \gamma_5\psi_{\mathrm{R}} = +\psi_{\mathrm{R}}, \\ \psi_{\mathrm{L}} &= \tfrac{1}{2}\,(1-\gamma_5)\psi, \quad \gamma_5\psi_{\mathrm{L}} = -\psi_{\mathrm{L}}\,. \end{aligned} \tag{3.118}$$

From (91) and (117), their Fourier series can be written as

$$\begin{aligned} \psi_{\mathrm{R}} &= \sum_{\boldsymbol{p}} \left[\, \psi^{(+)}_{\boldsymbol{p},\,1/2} b(\boldsymbol{p},\,1/2) \; + \psi^{(-)}_{-\boldsymbol{p},\,1/2} d^\dagger(\boldsymbol{p},\,-1/2) \right]\,, \\ \psi_{\mathrm{L}} &= \sum_{\boldsymbol{p}} \left[\psi^{(+)}_{\boldsymbol{p},-\,1/2} b(\boldsymbol{p},\,-1/2) + \psi^{(-)}_{-\boldsymbol{p},-\,1/2} d^\dagger(\boldsymbol{p},\,1/2) \right]\,. \end{aligned} \tag{3.119}$$

According to (119), ψ_{R} destroys positive-helicity states of particle and creates negative-helicity states of antiparticle, whereas ψ_{L} destroys $h = -1/2$ states of particle and creates $h = +1/2$ states of antiparticle. Table 3.2 summarizes these results.

For a zero-mass particle, chirality is well defined and Lorentz-invariant, and so ψ can exist either as a left-handed state, ψ_{L}, or as a right-handed state, ψ_{R}. For a given momentum $\boldsymbol{p}$, a massless particle can have its spin oriented parallel or antiparallel to its direction of motion, and each state can be described by a two-component spinor. Indeed, when $m = 0$, the Dirac Hamiltonian $\hat{H} = -\mathrm{i}\gamma_0\boldsymbol{\gamma}\cdot\nabla$ involves only three matrices, namely, $\gamma_0\gamma_i$ for $i = 1, 2, 3$, which satisfy the algebra

$$\{\gamma_0\gamma_i,\, \gamma_0\gamma_j\} = 2\,\delta_{ij}, \quad i, j = 1, 2, 3\,. \tag{3.120}$$

These relations can be satisfied by the 2×2 Pauli matrices, obviating the need for a formulation in terms of four-component spinors. The situation becomes particularly transparent when γ_5 is diagonal, as in the Weyl or *chiral representation* of the γ_μ matrices defined by

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.121}$$

In this representation the Weyl spinors of chiralities $\gamma_5 = +1$ and $\gamma_5 = -1$ take respectively the forms

$$\psi_{\mathrm{R}} = \begin{pmatrix} \chi_{\mathrm{R}} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi_{\mathrm{L}} = \begin{pmatrix} 0 \\ \chi_{\mathrm{L}} \end{pmatrix}, \tag{3.122}$$

Table 3.2. Chirality and helicity

Spinors		Chirality γ_5	Helicity h
$\psi^{(+)}_{\boldsymbol{p},1/2}(x)$	$u(\boldsymbol{p},+1/2)$	$+1$	$1/2$
$\psi^{(-)}_{-\boldsymbol{p},1/2}(x)$	$v(\boldsymbol{p},-1/2)$	$+1$	$-1/2$
$\psi^{(+)}_{\boldsymbol{p},-1/2}(x)$	$u(\boldsymbol{p},-1/2)$	-1	$-1/2$
$\psi^{(-)}_{-\boldsymbol{p},-1/2}(x)$	$v(\boldsymbol{p},+1/2)$	-1	$1/2$

where χ_{R} and χ_{L} are two-component spinors. Dirac's equation then becomes a system of two uncoupled equations for two-component spinors:

$$\begin{aligned} \mathrm{i}\partial_0\chi_{\mathrm{R}} &= -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla\chi_{\mathrm{R}}\,, \\ \mathrm{i}\partial_0\chi_{\mathrm{L}} &= \mathrm{i}\boldsymbol{\sigma}\cdot\nabla\chi_{\mathrm{L}}\,. \end{aligned} \tag{3.123}$$

The Lagrangian for a Dirac particle with mass $m \neq 0$, given by (75), may be rewritten in terms of Weyl spinors:

$$\mathcal{L} = \overline{\psi}_{\mathrm{R}}\,\mathrm{i}\gamma^\mu\partial_\mu\,\psi_{\mathrm{R}} + \overline{\psi}_{\mathrm{L}}\,\mathrm{i}\gamma^\mu\partial_\mu\,\psi_{\mathrm{L}} - m(\overline{\psi}_{\mathrm{R}}\,\psi_{\mathrm{L}} + \overline{\psi}_{\mathrm{L}}\,\psi_{\mathrm{R}})\,. \tag{3.124}$$

The components of opposite helicities are connected in the mass term and both therefore necessarily appear for a fermion of nonvanishing mass. However, when $m = 0$, the Lagrangian breaks up into two independent parts, one for each chirality,

$$\mathcal{L} = \chi^\dagger_{\mathrm{R}}\,\mathrm{i}\sigma^\mu\partial_\mu\chi_{\mathrm{R}} + \chi^\dagger_{\mathrm{L}}\,\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_{\mathrm{L}}\,, \tag{3.125}$$

where

$$\sigma^\mu = (1,\boldsymbol{\sigma})\,, \qquad \bar{\sigma}^\mu = (1,-\boldsymbol{\sigma})\,. \tag{3.126}$$

This decomposition into left- and right-handed states not only simplifies the formalism but also turns out to be a necessity because it is now known that only left-handed neutrinos and right-handed antineutrinos exist and they can be described naturally in terms of the Weyl spinors. Right-handed neutrinos, even if they exist, are not observed in weak interaction reactions, are not coupled to known particles, and cannot acquire mass through interactions. Therefore, models of weak interactions will involve only left-handed neutrinos and right-handed antineutrinos.

Problems

3.1 Boosting a fermion from rest. The Dirac spinor for a free particle of momentum $\boldsymbol{p}$ can be obtained from the corresponding solution for $\boldsymbol{p} = 0$ by a Lorentz boost. As an example, calculate $u(\boldsymbol{p}, s) = S_{\mathrm{L}}(\omega)\sqrt{2m}\, u(\mathbf{0}, s)$, where $\omega_{\mu\nu}$ give the boost parameters.

3.2 Γ matrices. (a) Prove that Γ_i, i =S, V, T, A, P, satisfy the conjugation property, $\gamma_0 \Gamma_i^\dagger \gamma_0 = \Gamma_i$, and produce 16 linearly independent matrices. (b) Show that two sets γ_μ and γ'_μ satisfying the relation $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ are related by $\gamma'_\mu = S\gamma_\mu S^{-1}$ for some 4×4 matrix S [for help, consult Good, R. H., Rev. Mod. Phys. **27** (1955) 187].

3.3 Bilinear covariants. Prove that the bilinear covariants given in Table 3.1 are Hermitian, and satisfy the Lorentz transformation properties shown in the table.

3.4 Majorana and Weyl representations. (a) Find the matrices S that transform the Pauli–Dirac standard representation of the γ-matrices into their Majorana and the Weyl representations:

$$\gamma_\mu^{(\mathrm{M})} = S_{\mathrm{M}} \gamma_\mu S_{\mathrm{M}}^{-1} \,,$$
$$\gamma_\mu^{(\mathrm{W})} = S_{\mathrm{W}} \gamma_\mu S_{\mathrm{W}}^{-1} \,,$$

where the Majorana representation is defined by

$$\gamma^1 = \begin{pmatrix} \mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3 \end{pmatrix} , \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \quad \gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix} ,$$
$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} ;$$

and the Weyl representation by

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

(b) Find the analogs of the spinors (45) and (46) in the Majorana and Weyl representations of the γ_μ matrices.

3.5 Orthogonality of spinors. Prove the following relations:
(a) $v^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}, s') = u^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s') = \delta_{ss'}\, 2\boldsymbol{\sigma} \cdot \boldsymbol{p}/(E + m)\,,$
(b) $v^\dagger(-\boldsymbol{p}, s) u(\boldsymbol{p}, s) = 0$.

3.6 Closure relation. Let $\psi(\boldsymbol{p})$ be an arbitrary spinor of momentum p_μ given by the sum

$$\psi(\boldsymbol{p}) = \sum_s \left[A_s u(\boldsymbol{p}, s) + B_s v(-\boldsymbol{p}, s) \right] .$$

Calculate the coefficients A_s and B_s, and prove the closure relation (55) .

3.7 Gordon identities. Let ψ_1 and ψ_2 be solutions to the Dirac equations $(\mathrm{i}\gamma^\mu \partial_\mu - m_i)\psi_i(x) = 0$. Prove the following relations:

$$(m_1 + m_2)\psi_2\gamma_\mu\psi_1 = \overline{\psi}_2 \left(-\mathrm{i}\overleftarrow{\partial}_\mu + \mathrm{i}\overrightarrow{\partial}_\mu \right) \psi_1 + \partial^\nu \left(\overline{\psi}_2 \sigma_{\mu\nu} \psi_1 \right) ,$$

$$(m_1 + m_2)\psi_2\gamma_\mu\gamma_5\psi_1 = (-\mathrm{i}\partial_\mu) \left(\overline{\psi}_2 \gamma_5 \psi_1 \right) + \overline{\psi}_2 \left(-\mathrm{i}\overleftarrow{\partial}_\nu + \mathrm{i}\overrightarrow{\partial}_\nu \right) \sigma_\mu{}^\nu \gamma_5 \psi_1 .$$

3.8 Fierz transformation. As the 16 Γ^i matrices in Problem 3.2, with i =S, V, T, A, P, form a complete set of $N = 4$ matrices, any product of bilinear covariants of the form $(\bar{u}_1\Gamma^i u_2)(\bar{u}_3\Gamma^j u_4)$ can be expressed as a linear combination of similar products written with a different sequence of spinors

$$(\bar{u}_1\Gamma^i u_2)(\bar{u}_3\Gamma^j u_4) = \sum_{mn} C^{ij}_{mn} (\bar{u}_1\Gamma^m u_4)(\bar{u}_3\Gamma^n u_2) .$$

In general, the spinors u_i refer to different particles. Show that

$$C^{ij}_{mn} = \frac{1}{N_m N_n} \mathrm{Tr}\,(\Gamma^i\Gamma^n\Gamma^j\Gamma^m).$$

Γ^i are assumed to be orthonormalized such that $\mathrm{Tr}\,(\Gamma^i\Gamma^j) = N_i\delta_{ij}$.

3.9 The Dirac Lagrangian. Show that the Lagrangian

$$\mathcal{L}_F = \overline{\psi}\,[\tfrac{1}{2}\,\mathrm{i}\gamma^\mu \overleftrightarrow{\partial}_\mu - m]\,\psi$$

differs from (75) only by a total derivative and therefore leads to the same equation of motion.

3.10 Spin of hyperon Λ. Λ^0 particles are produced in the reaction $\pi^- + \mathrm{p} \to \mathrm{K}^0 + \Lambda^0$ and are identified by their subsequent disintegrations $\Lambda \to \pi^- + \mathrm{p}$. Assuming that the proton spin ($1/2$) and the π, K spins (0) are known, we want to determine the spin of Λ^0. (a) For K and Λ produced in the direction of the incoming beam (parallel to the z axis chosen as the axis of quantization), what are the possible values of S_z, the z component of the Λ spin? (b) For polarized protons, show that in the Λ rest frame the angular distributions of Λ produced in the incident beam direction are given for different values of Λ by

$$\begin{aligned} S_\Lambda &= 1/2 && \text{isotropic}, \\ S_\Lambda &= 3/2 && (1 + 3\cos^2\theta), \\ S_\Lambda &= 5/2 && (1 - 2\cos^2\theta + 5\cos^4\theta)\,, \end{aligned}$$

where θ is the relative angle of the Λ disintegration products. (The observed distributions turn out to be isotropic, indicating that the Λ spin is most likely equal to $1/2$.)

3.11 Anticommutation relations between Dirac fields. Let $\psi_a(x)$, with $a = 1, \ldots, 4$, be the components of the Dirac spinor of a free particle of mass m.
(a) Using the completeness of spinors and (98), show that for $x_0 = y_0$,

$$\{\psi_a(x), \psi_b^\dagger(y)\}|_{x^0=y^0} = \delta_{ab}\,\delta(\boldsymbol{x} - \boldsymbol{y})\,.$$

(b) Using projection operators and (98), show that for arbitrary x and y,

$$\{\psi_a(x),\, \psi_b^\dagger(y)\} = -\mathrm{i}S_{ab}(x - y, m),$$

where

$$\begin{aligned} S(x, m) &= -(\mathrm{i}\gamma \cdot \partial + m)\,\Delta(x, m) \\ &= -(\mathrm{i}\gamma \cdot \partial + m)\,\frac{-\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4p\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,\delta(p^2 - m^2)\,\epsilon(p_0)\,. \end{aligned}$$

Here $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = 0$ for $x < 0$.

Suggestions for Further Reading

The classics:

Dirac, P. A. M., Proc. Roy. Soc. (London) **A117** (1928) 610

Majorana, E., Nuovo Cimento **14** (1937) 171

Connections between spins and statistics:

Lüders, C., Ann. Phys. **2** (1957) 1

Pauli, W., Phys. Rev. **58** (1940) 716

Nonrelativistic limit of Dirac's equation:

Foldy, L. L. and Wouthuysen, S. A., Phys. Rev. **78** (1950) 29